Information Rates for Phase Noise Channels (including fiber optic channels)

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Claude Elwood Shannon
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1) Phase Noise Models

- Phase noise due to (1) oscillator instability; (2) fiber non-linearities
- Phase noise statistics:
  - phase-locked loops (PLLs) *residual* noise: von Mises/Tikhonov distribution
  - satellite (DVB-S2): white Gaussian process *filtered* by IIR filters
  - fiber-optic lasers: *Wiener* process
  - Raman amplification: *large bandwidth* Gaussian process
White Phase Noise

- Simplified model (Barletta-Kramer, 2014)

\[ R(t) = X(t) \cdot e^{j\Theta(t)} + N(t) \]

\( \Theta(t) \) is white* and \( N(t) \) is white Gaussian* (both are idealizations)

- Motivation: phase noise bandwidth much larger than receiver bandwidth
- Mathematically: let \( \{\varphi_m(t)\} \) be an orthonormal basis of \( L^2[0,T] \) and project \( X(t), N(t), \) and \( R(t) \) onto the \( \varphi_m(t) \)

* We use \( E[\Theta(t)\Theta(t+\tau)] = \sigma^2\delta(\tau) \) and \( E[N(t)N(t+\tau)] = \sigma^2\delta(\tau) \)
Discretization (1)

- **X(t) and N(t):**
  \[ X(t) = \sum_{m=1}^{M} X_m \phi_m(t) \quad N(t) = \sum_{m=1}^{\infty} N_m \phi_m(t) \]

- **Receiver:**
  \[
  Y_k = \left\langle X(t) e^{j\Theta(t)} + N(t), \phi_k(t) \right\rangle \\
  = \sum_{m=1}^{M} X_m \left\langle \phi_m(t) e^{j\Theta(t)}, \phi_k(t) \right\rangle + N_k
  \]
  \[
  \Phi_{m,k}
  \]
Discretization (2)

Samples:

\[
\Phi_{m,k} = \int_0^T \phi_m(t) e^{j\theta(t)} \phi_k(t)^* \, dt
\]

\[
= \lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^L \phi_m(t_i^{(L)}) e^{j\theta(t_i^{(L)})} \phi_k(t_i^{(L)})^*, \quad t_i^{(L)} = \frac{(i - 1)T}{L}
\]

\[
E[e^{j\theta(t)}], \quad m = k
\]

\[
0, \quad \text{else}
\]

Barletta-Kramer, 2014: Almost sure convergence for white phase noise with uncorrelated samples of process \{e^{j\theta(t)}\}
Discretization (3)

Model*: \( Y_k = X_k \cdot E[e^{j\Theta(t)}] + N_k \)

An AWGN channel (!) but with SNR penalty \( |\mu_{\Theta}|^2 \)

Penalty called spectral loss**: “lost” power is spread across all frequencies as white noise

Proof: use Borel-Cantelli lemma with a classic trick and a simplified step via assumed boundedness of \( \int |\phi_m(t) \phi_k(t)^*| \, dt \)

I expect this insight to be useful for fiber channels

* Barletta-Kramer 2014, ** Goebel et al. 2011
2) Fiber Channel(s)

- Single-Mode Fiber (SMF): a small core that carries one mode of light
- Here one mode has 2 complex dimensions: two polarizations
- Theory papers often consider one complex dimension; the general case is interesting too of course (see below)
- In fact, a hot topic in the fiber community is MIMO fiber

![Single-mode fiber (SMF)](image)
![Multi-mode fiber (MMF)](image)
![Multi-core fiber (MCF)](image)
Maxwell’s equations and low-order approximations result in a generalized nonlinear Schrödinger equation (GNSE):

\[
\frac{\partial E}{\partial z} = -\frac{\alpha}{2}E - \frac{i}{2} \beta_2 \frac{\partial^2 E}{\partial T^2} + \frac{1}{6} \beta_3 \frac{\partial^3 E}{\partial T^3} + i\gamma |E|^2 E + n
\]

- \( E \): Electromagnetic field, function of \( z \) and \( T \)
- \( z \): Distance
- \( T \): Retarded time \( t-\beta_1 z \)
- \( \alpha \): Fiber loss coefficient (~ 3 dB/15 km)
- \( \beta_1 \): Inverse of group velocity
- \( \beta_2 \): Fiber group velocity dispersion
- \( \beta_3 \): Fiber dispersion slope (include if \( \beta_2 \) small)
- \( \gamma \): Fiber nonlinear parameter \( (n_2 \omega)/(cA_{\text{eff}}) \)
- \( n_2 \): Fiber nonlinear coefficient
- \( \omega \): Angular frequency
- \( c \): Speed of light
- \( A_{\text{eff}} \): Fiber effective area

• To simulate, split the fiber length $z^*$ into $K$ small steps ($\Delta z$) and the time $T$ into $L$ small steps ($\Delta t$)

• Split-step Fourier method at distance $z_k$, $k=0,1,...,K$

- Ideal Raman amplification: removes the loss and adds noise
- $F = \text{Fourier transform}$
- $D_L = \text{diagonal matrix with fixed entries of unit amplitude}$ (all-pass filter)
- $D_N = \text{diagonal matrix with unit amplitude entries}$; the $(\ell,\ell)$-entry phase shift is proportional to the magnitude-squared of the $\ell^{th}$ entry of $E_N(z_{k+1})$
Consider a complex column vector $\mathbf{X} = \mathbf{X}_c + j \mathbf{X}_s$ with covariance and pseudo-covariance matrices

\[
\mathbf{Q}_\mathbf{X} = E\left[ (\mathbf{X} - E[\mathbf{X}]) (\mathbf{X} - E[\mathbf{X}])^\dagger \right]
\]

\[
\tilde{\mathbf{Q}}_\mathbf{X} = E\left[ (\mathbf{X} - E[\mathbf{X}]) (\mathbf{X} - E[\mathbf{X}])^T \right]
\]

For interest: $\mathbf{X}$ is called proper if its pseudo-covariance matrix is 0.

Example: Consider a complex, zero-mean, scalar $\mathbf{X} = \mathbf{X}_c + j \mathbf{X}_s$. $\mathbf{X}$ is proper if $E[\mathbf{X}_c^2]=E[\mathbf{X}_s^2]$ and $E[\mathbf{X}_c \mathbf{X}_s]=0$.

Note: circularly symmetric $\mathbf{X}$ are proper, but proper $\mathbf{X}$ are not necessarily circularly symmetric (e.g. QAM signal sets)
Maximum Entropy

- **Maximum Entropy**: consider the correlation matrix $R_X = E[X X^\dagger]$ where $X$ has $L$ entries. Then

$$h(X) \leq \log \left[ (\pi e)^L \det R_X \right]$$

with equality if and only if $X$ is Gaussian and proper (or circularly symmetric).

- For a complex square matrix $M$ we have

$$h(MX) = h(X) + 2 \log |\det (M)|$$

In particular, if $M$ is unitary then $h(MX) = h(X)$
Entropic Power Inequality

- Entropy Power:
  \[ V(X) = e^{h(X)/L} \sqrt{\pi e} \]  

- Entropy Power Inequality: for independent \( X \) and \( Y \) we have
  \[
  V(X + Y) \geq V(X) + V(Y)
  \]

- Conditional version: for conditionally independent \( X \) and \( Y \) we have
  \[
  V(X | U) = e^{h(X | U)/L} \sqrt{\pi e} \\
  V(X + Y | U) \geq V(X | U) + V(Y | U)
  \]
Main Observations

- The linear step conserves energy and entropy
- The non-linear step also conserves energy and entropy (the key result)

\[
h\left(|a| e^{j \text{arg}(a) + jf(|a|)}\right) = h\left(|a|, \text{arg}(a) + f(|a|)\right) + E[\log |a|]
\]

\[
= h(|a|) + h\left(\text{arg}(a) + f(|a|) | |a|) + E[\log |a|] = h(a)
\]

\[
h\left(|a|, \text{arg}(a)\right)
\]
Energy Recursion

\[ E(z_k) \xrightarrow{F} D_L \xrightarrow{F^{-1}} D_N \xrightarrow{\text{Noise}} E(z_{k+1}) \]

**Linear**

**Nonlinear**

- **Energy** after K steps: \( \text{Energy}_{\text{Launch}} + KN \). We thus have:

\[
\begin{align*}
    h(E(z_K)) &\leq \log \left( (\pi e)^L \det \left( R(E(z_K)) \right) \right) \quad \ldots \text{maximum entropy} \\
    \leq \sum_{i=1}^{L} \log \left( \pi e R_{i,j}(E(z_K)) \right) \quad \ldots \text{Hadamard's inequality} \\
    \leq L \cdot \log \left( \pi e (\text{Energy}_{\text{Launch}} + KN)/L \right) \quad \ldots \text{Jensen's inequality}
\end{align*}
\]
Entropy Recursion

- **Entropy recursion:**
  
  \[
  V\left(E(z_{k+1})|E(z_0)\right) \geq V\left(E(z_k)|E(z_0)\right) + N/L
  \]

- We thus have:
  
  \[
  V\left(E(z_K)|E(z_0)\right) \geq KN/L
  \]

  or

  \[
  h\left(E(z_K)|E(z_0)\right) \geq L \log(\pi e KN/L)
  \]
So for every step we have:

- **Signal energy** grows by the noise variance: can *upper* bound $h(\mathbb{E}(z_K))$
- **Entropy power** grows by at least the noise variance:
  can *lower* bound $h(\mathbb{E}(z_K | \mathbb{E}(z_0))$
- **Result**: 

\[
I(\mathbb{E}(z_0);\mathbb{E}(z_K)) = h(\mathbb{E}(z_K)) - h(\mathbb{E}(z_K | \mathbb{E}(z_0))) \\
\leq L \cdot \log(1 + \text{SNR})
\]

*SNR = receiver signal-to-noise ratio*
\[ \Rightarrow \frac{1}{L} I(E(z_0); E(z_K)) \leq \log(1 + SNR) \]

- Let \( B = 1/\Delta t \) be the “bandwidth” of the simulation
- So \( L = T/\Delta t = TB \) is the time-bandwidth product
- The spectral efficiency is thus bounded by

\[ \eta \leq \log(1 + SNR) \] [bits/sec/Hz]
5) Conclusions

1) Spectral efficiency of (an idealized model of) SMF with linear polarization is $\leq \log(1+\text{SNR})$

2) Many extensions are possible:
   - lumped amplification, 3rd-order dispersion, delayed Kerr effect
   - uniform loss, linear filters (for capacity results)
   - MIMO fiber (MMF or MCF)

3) More difficult:
   - better bounds and understanding at high SNR
   - frequency-dependent loss, dispersion, non-linearity