Capacity Achieving Codes: There and Back Again

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Outline

Introduction

Factor Graphs

Message Passing

Applications of Factor Graphs

Applications of EXIT Curves

Spatially-Coupled Factor Graphs

Universality for Multiuser Scenarios

Abstract Formulation of Threshold Saturation
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Capacity of Point-to-Point Communication

- Coding for Discrete-Time Memoryless Channels
  - Transition probability: $P_{Y|X}(y|x)$ for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$
  - Transmit a length-$n$ codeword $\underline{x} \in \mathcal{C} \subset \mathcal{X}^n$
  - Decode to most likely codeword given received $\underline{y}$
Capacity Achieving Codes: There and Back Again

Capacity of Point-to-Point Communication

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- **Channel Capacity** introduced by Shannon in 1948
  - Random code of rate $R \triangleq \frac{1}{n} \log_2 |\mathcal{C}|$ (bits per channel use)
  - As $n \to \infty$, reliable transmission possible if $R < C$ with

$$C \triangleq \max_{p(x)} I(X; Y)$$
The Binary Erasure Channel (BEC)

- Denoted BEC(ε) when erasure probability is ε
- \( C = 1 - \varepsilon \) = expected fraction bits not erased
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- Coding with a binary linear code
  - Parity-check matrix \(H \in \{0, 1\}^{m \times n}\) with \(m = (1 - R)n\)
  - Codebook \(C \triangleq \{x \in \{0, 1\}^n \mid Hx = 0\}\) has \(2^{Rn}\) codewords
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Let \(\mathcal{E}\) denote the index set of erased positions so that

\[
H_x = \begin{bmatrix} H_{\mathcal{E}} & H_{\mathcal{E}c} \end{bmatrix} \begin{bmatrix} x_{\mathcal{E}} \\ y_{\mathcal{E}c} \end{bmatrix} = 0 \iff H_{\mathcal{E}}x_{\mathcal{E}} = -H_{\mathcal{E}c}y_{\mathcal{E}c}
\]

- Decoding fails iff: \(H_{\mathcal{E}}\) singular \(\iff\) cw exists with 1’s only in \(\mathcal{E}\)
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  \]
- Decoding fails iff: $H_E$ singular $\iff$ cw exists with 1’s only in $E$
- One can achieve capacity by drawing $H$ uniformly at random!
Some Early Milestones in Coding

- 1948: Shannon defines channel capacity and random codes
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- 1955: Elias introduces the erasure channel and convolutional codes; also shows random parity-check codes achieve capacity on the BEC
- 1959: BCH Codes (Hocquenghem’59 and Bose-Ray-Chaudhuri’60)
- 1960: Gallager introduces low-density parity-check (LDPC) codes and iterative decoding
- 1960: Reed-Solomon codes
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But, more than 35 years passed before we could:

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- Provably achieve capacity with deterministic constructions
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- 1999-2011: Understanding LDPC convolutional codes and coupling
Key Tools That Made the Difference

- Factor Graph (FG)
  - Compact description of joint distribution for random variables
  - Natural setup for inference problems with partial observations
  - Belief-Propagation (BP)
    - Message-passing algorithm for inference on a FG
    - Probability estimates are passed along edges in the factor graph
    - Provides exact marginals if factor graph is a tree
  - Density Evolution (DE)
    - Tracks distribution of messages passed by belief propagation
    - In some cases, allows rigorous analysis of BP-based inference
  - EXtrinsic Information Transfer (EXIT) Curves
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Applications of These Tools

- Error-Correcting Codes
  - Random code defined by random factor graph
  - Low-complexity decoding via belief propagation
  - Analysis of belief-propagation decoding via density evolution
  - Provides code constructions that provably achieve capacity!
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- **Boolean Satisfiability: K-SAT**
  - Random instance of K-SAT defined by random factor graph
  - Non-rigorous analysis via the cavity method
  - Predicted thresholds later proved exact!
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- Compressed Sensing
  - Random measurement matrix defined by random factor graph
  - Low-complexity reconstruction via message passing
  - Schemes provably achieve the information-theoretic limit!
Polya’s Dictum

If you can’t solve a problem, then it probably contains an easier problem that you can’t solve: find it.
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If you can’t solve a problem, then it probably contains an easier problem that you can’t solve: find it.

- The solution of the simpler problem often provides insight that allows one to crack the harder problem.

- To achieve channel capacity in practice, we now know that a good “easy” problem would have been:
  - “Design a code that achieves capacity on the BEC and is encodable and decodable in quasi-linear time”
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Factor Graphs

- A factor graph provides a **graphical representation** of the **local dependence structure** for a set of random variables.
  - Bipartite graph with variables $x_1, \ldots, x_n$ and factors $f_1, \ldots, f_m$.

Consider random variables $(X_1, X_2, X_3, X_4) \in \mathcal{X}$ and $Y$ where:

$$
P(x_1, x_2, x_3, x_4) \doteq P(X_1 = x_1, X_2 = x_2, \ldots, X_4 = x_4 | Y = y) \propto f(x_1, x_2) f_2(x_2, x_3) f_3(x_3, x_4)$$

Given $Y = y$, this describes a Markov chain whose factor graph is:

\[x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} x_3 \xrightarrow{f_3} x_4\]
Factor Graphs

- A factor graph provides a **graphical representation** of the **local dependence structure** for a set of random variables.
  - Bipartite graph with variables $x_1, \ldots, x_n$ and factors $f_1, \ldots, f_m$

- Consider random variables $(X_1, X_2, \ldots, X_4) \in \mathcal{X}^4$ and $Y$ where:

\[
P(x_1, x_2, x_3, x_4) \triangleq \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots, X_4 = x_4 | Y = y)
\propto f(x_1, x_2, x_3, x_4)
\triangleq f_1(x_1, x_2) f_2(x_2, x_3) f_3(x_3, x_4)\]
Factor Graphs

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![Factor Graph Diagram](attachment:image.png)
Conditional Independence for Factor Graphs

- Let $A, B, S \subset [n]$ be disjoint subsets of VNs in factor graph $G$
  - If $S$ separates $A$ from $B$ (i.e., there is no path in $G$ from $A$ to $B$ that avoids $S$), then we have $X_A \perp \perp X_B \mid X_S$

$$P(x_A, x_B \mid x_S) = P(x_A \mid x_S)P(x_B \mid x_S)$$
conditional independence for factor graphs

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- Markov chain example: $A = \{x_1, x_2\}$, $B = \{x_4\}$, $S = \{x_3\}$

```
  x1 ------ f12 ------ x2 ------ f23 ------ x3 ------ f34 ------ x4
```

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Sketch of Proof:
Fixing $X_S = x_S$ separates the FG into disjoint components.
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- Sketch of Proof:
  - Fixing $X_S = x_S$ separates the FG into disjoint components
  - Groups of VNs in different components are independent
  - $X_A \perp \perp X_B$ because $A$ and $B$ are in different components
Inference via Marginalization

- Marginalizing out all variables except $X_1$ gives

$$\mathbb{P}(X_1 = x_1 | Y = y) \propto g_1(x_1) \triangleq \sum_{(x_2, \ldots, x_4) \in \mathcal{X}^3} f(x_1, x_2, x_3, x_4)$$

- Thus, the maximum a posteriori decision for $X_1$ given $Y = y$ is

$$\hat{x}_1 = \arg \max_{x_1 \in \mathcal{X}} \sum_{(x_2, \ldots, x_4) \in \mathcal{X}^3} f(x_1, x_2, x_3, x_4)$$

- For a general function, this requires roughly $|\mathcal{X}|^4$ operations
Inference via Marginalization

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- **For a general function,** this requires roughly $|\mathcal{X}|^4$ **operations**

- **Marginalization** is efficient for **tree-structured factor graphs**

- **For the Markov chain,** roughly $5 |\mathcal{X}|^2$ **operations** required

$$g_1(x_1) = \sum_{x_2 \in \mathcal{X}} f_1(x_1, x_2) \sum_{x_3 \in \mathcal{X}} f_2(x_2, x_3) \sum_{x_4 \in \mathcal{X}} f_3(x_3, x_4)$$
Consider a random vector \((X_1, X_2, \ldots, X_6) \in \mathcal{X}^6\) where

\[
\mathbb{P}(X_1 = x_1, \ldots, X_6 = x_6 | Y = y) \propto f(x_1, x_2, x_3, x_4, x_5, x_6)
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Consider a random vector \((X_1, X_2, \ldots, X_6) \in \mathcal{X}^6\) where
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\]

Brute force marginal requires \(|\mathcal{X}|^5\) operations for each \(x_1 \in \mathcal{X}\):
\[
g_1(x_1) \triangleq \sum_{x_2^6 \in \mathcal{X}^5} f(x_1, x_2, x_3, x_4, x_5, x_6)
\]

Thus, we need \(|\mathcal{X}|^6\) operations
The Importance of Factorization (1)

- Consider a random vector \((X_1, X_2, \ldots, X_6) \in \mathcal{X}^6\) where

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\]

- Thus, we need \(|\mathcal{X}|^6\) operations

- If \(f\) factors as follows, then the marginalization can be simplified:

\[
f(x_1, x_2, x_3, x_4, x_5, x_6) = f_1(x_1, x_2, x_3)f_2(x_1, x_4, x_6)f_3(x_4)f_4(x_4, x_5)
\]
The Importance of Factorization (2)

For example, we can write $g_1(x_1)$ as:

$$= \sum_{x_2^6} f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5)$$
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$$
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$$= \sum_{x_2^4} f_1(x_1, x_2, x_3) f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right]$$

This implementation requires roughly $2|X|^3 + 5|X|^2$ operations.
The Importance of Factorization (2)

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\[
\begin{align*}
&= \sum_{x_2^6} f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5) \\
&= \sum_{x_2^5} f_1(x_1, x_2, x_3) f_3(x_4) f_4(x_4, x_5) \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right] \\
&= \sum_{x_2^4} f_1(x_1, x_2, x_3) f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right] \\
&= \sum_{x_2^3} f_1(x_1, x_2, x_3) \left[ \sum_{x_4} f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right] \right]
\end{align*}
\]

This implementation requires roughly \( 2|X|^3 + 5|X|^2 \) operations.
The Importance of Factorization (2)

For example, we can write $g_1(x_1)$ as:

$$= \sum_{x_2^6} f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5)$$

$$= \sum_{x_2^5} f_1(x_1, x_2, x_3) f_3(x_4) f_4(x_4, x_5) \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right]$$

$$= \sum_{x_2^4} f_1(x_1, x_2, x_3) f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right]$$

$$= \sum_{x_2^3} f_1(x_1, x_2, x_3) \left[ \sum_{x_4} f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right] \right]$$

$$= \sum_{x_2} \left[ \sum_{x_3} f_1(x_1, x_2, x_3) \right] \left[ \sum_{x_4} f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] \left[ \sum_{x_6} f_2(x_1, x_4, x_6) \right] \right]$$

This implementation requires roughly $2 |\mathcal{X}|^3 + 5 |\mathcal{X}|^2$ operations.
The Factor Graph and Leaf Removal

\[ g_1(x_1) = \sum_{x_2^5} f_1(x_1, x_2, x_3) f_3(x_4) f_4(x_4, x_5) \sum_{x_6} f_2(x_1, x_4, x_6) \]
The Factor Graph and Leaf Removal

\[
g_1(x_1) = \sum_{x_2^4} f_1(x_1, x_2, x_3) f_3(x_4) \left[ \sum_{x_5} f_4(x_4, x_5) \right] f_2'(x_1, x_4)
\]
The Factor Graph and Leaf Removal

\[ g_1(x_1) = \sum_{x_2^3} f_1(x_1, x_2, x_3) \left[ \sum_{x_4} f_3(x_4) f_4'(x_4) f_2'(x_1, x_4) \right] \]
The Factor Graph and Leaf Removal

\[ g_1(x_1) = \sum_{x_2} \left[ \sum_{x_3} f_1(x_1, x_2, x_3) \right] f''_2(x_1) \]
The Factor Graph and Leaf Removal

\[ g_1(x_1) = \left[ \sum_{x_2} f'_1(x_1, x_2) \right] f''_2(x_1) \]
$g_1(x_1) = f''_1(x_1)f''_2(x_1)$
A non-negative function $f : \mathcal{X}^n \to \mathbb{R}$ defines a distribution on $\mathcal{X}^n$:

$$P(x) \triangleq \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \frac{1}{Z} f(x) \triangleq \frac{1}{Z} \prod_{a=1}^{m} f_a(x_{\partial a}),$$

where $x_{\partial a}$ is the subvector of variables involved in factor $a$ and $Z \triangleq \sum_x f(x)$ is called the partition function.
Constraint Satisfaction and Zero-One Factors

- A non-negative function \( f : \mathcal{X}^n \rightarrow \mathbb{R} \) defines a distribution on \( \mathcal{X}^n \):

\[
P(x) \triangleq \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)
= \frac{1}{Z} f(x) \triangleq \frac{1}{Z} \prod_{a=1}^{m} f_a(x_{\partial a}),
\]

- where \( x_{\partial a} \) is the subvector of variables involved in factor \( a \)
- and \( Z \triangleq \sum_x f(x) \) is called the partition function

- For Constraint Satisfaction Problems (CSPs)
  - All factors \( f_a(x_{\partial a}) \) take values in \( \{0, 1\} \)
  - The set of valid configurations is \( \{x \in \mathcal{X}^n | f(x) = 1\} \)
  - Thus, \( Z \) equals the number of valid configurations
  - \( P(x) \) is uniform over the set of valid configurations
Marginalization via Belief Propagation

- Factor Graph $G = (V \cup F, E)$
  - Variable nodes $V$, Factor nodes $F$
  - Edges: $(i, a) \in E \subseteq V \times F$
  - $F(i) / V(a) = \text{set of neighbors for node-}i/a$
  - Messages: $\mu_{i \rightarrow a}(x_i)$ and $\hat{\mu}_{a \rightarrow i}(x_i)$
Marginalization via Belief Propagation

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  - Messages: $\mu_{i\rightarrow a}^{(t)}(x_i)$ and $\hat{\mu}_{a\rightarrow i}^{(t)}(x_i)$

- **variable-i to factor-a message**

$$
\hat{\mu}_{b_1\rightarrow i}^{(t)}(x_i) \\
\hat{\mu}_{b_2\rightarrow i}^{(t)}(x_i) \\
\hat{\mu}_{b_3\rightarrow i}^{(t)}(x_i)

\mu_{i\rightarrow a}^{(t+1)}(x_i) = \prod_{b \in F(i) \setminus a} \hat{\mu}_{b\rightarrow i}^{(t)}(x_i)$$
Marginalization via Belief Propagation

▶ Factor Graph $G = (V \cup F, E)$
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  - Edges: $(i, a) \in E \subseteq V \times F$
  - $F(i)/V(a)$ = set of neighbors for node-$i/a$
  - Messages: $\mu_{i \rightarrow a}^{(t)}(x_i)$ and $\hat{\mu}_{a \rightarrow i}^{(t)}(x_i)$

▶ factor-$a$ to variable-$i$ message

$\mu_{j_1 \rightarrow a}^{(t)}(x_{j_1})$

$\mu_{j_2 \rightarrow a}^{(t)}(x_{j_2})$

$\mu_{j_3 \rightarrow a}^{(t)}(x_{j_3})$

$a$ → $\hat{\mu}_{a \rightarrow i}^{(t)}(x_i) = \sum_{x_{V(a)} \setminus i} f_a(x_{V(a)}) \prod_{j \in V(a) \setminus i} \mu_{j \rightarrow a}^{(t)}(x_j)$
Factor Graph $G = (V \cup F, E)$

- Variable nodes $V$, Factor nodes $F$
- Edges: $(i, a) \in E \subseteq V \times F$
- $F(i)/V(a) =$ set of neighbors for node-$i/a$
- Messages: $\mu_{i \rightarrow a}(x_i)$ and $\hat{\mu}_{a \rightarrow i}(x_i)$

variable-$i$ marginal

$$\hat{\mu}_{b_1 \rightarrow i}(x_i)$$
$$\hat{\mu}_{b_2 \rightarrow i}(x_i)$$
$$\hat{\mu}_{b_3 \rightarrow i}(x_i)$$

$$\hat{\mu}_{b_4 \rightarrow i}(x_i)$$

$$\mu_{i}^{(t+1)}(x_i) = \prod_{b \in F(i)} \hat{\mu}_{b \rightarrow i}(x_i)$$
Marginalization via Belief Propagation: Example

iteration 1: variable to factor

\[ \mu_{i \rightarrow a}^{(1)}(x_i) = 1 \]
Marginalization via Belief Propagation: Example

iteration 1: variable to factor

\[ \mu_{i \rightarrow a}^{(1)}(x_i) = 1 \]

iteration 1: factor to variable

\[ \hat{\mu}_{4 \rightarrow 4}^{(1)}(x_4) = \sum_{x_5} f_4(x_4, x_5) \mu_{5 \rightarrow 4}^{(1)}(x_i) \]

\[ = \sum_{x_5} f_4(x_4, x_5) \]

\[ \hat{\mu}_{3 \rightarrow 4}^{(1)}(x_4) = f_3(x_4) \]
Marginalization via Belief Propagation: Example

iteration 1: factor to variable

\[ \hat{\mu}_{4 \rightarrow 4}^{(1)}(x_4) = \sum_{x_5} f_4(x_4, x_5) \mu_{5 \rightarrow 4}^{(1)}(x_i) \]

\[ = \sum_{x_5} f_4(x_4, x_5) \]

\[ \hat{\mu}_{3 \rightarrow 4}^{(1)}(x_4) = f_3(x_4) \]

iteration 2: variable to factor

\[ \mu_{4 \rightarrow 2}^{(2)}(x_4) = \hat{\mu}_{4 \rightarrow 4}^{(1)}(x_4) \hat{\mu}_{3 \rightarrow 4}^{(1)}(x_4) \]

\[ = f_3(x_4) \sum_{x_5} f_4(x_4, x_5) \]

\[ \mu_{6 \rightarrow 2}^{(2)}(x_6) = 1 \]
Marginalization via Belief Propagation: Example

iteration 2: variable to factor

\[ \mu_{4\rightarrow2}^{(2)}(x_4) = \hat{\mu}_{4\rightarrow4}(x_4)\hat{\mu}_{3\rightarrow4}(x_4) = f_3(x_4) \sum_{x_5} f_4(x_4, x_5) \]

\[ \mu_{6\rightarrow2}^{(2)}(x_6) = 1 \]

iteration 2: factor to variable

\[ \hat{\mu}_{2\rightarrow1}^{(2)}(x_1) = \sum_{x_4, x_6} f_2(x_1, x_4, x_6)\mu_{4\rightarrow2}^{(2)}(x_4)\mu_{6\rightarrow2}^{(2)}(x_6) = \sum_{x_4, x_6} f_2(x_1, x_4, x_6)f_3(x_4) \sum_{x_5} f_4(x_4, x_5) = f_2''(x_1) \]
Margarinalization via Belief Propagation: Example

iteration 2: variable marginal

\[ \mu_1^{(3)}(x_1) = \hat{\mu}_{1\to1}(x_1)\hat{\mu}_{2\to1}(x_1) = f''_1(x_1)f''_2(x_2) \]

Same answer as peeling but from a distributed parallel algorithm
Outline

Introduction

Factor Graphs

Message Passing

Applications of Factor Graphs

Applications of EXIT Curves

Spatially-Coupled Factor Graphs

Universality for Multiuser Scenarios

Abstract Formulation of Threshold Saturation
Sudoku: A Factor Graph for the Masses

rows are permutations of \(\{1, 2, \ldots, 9\}\)
Sudoku: A Factor Graph for the Masses

rows are permutations of \(\{1, 2, \ldots, 9\}\)
columns are permutations of \(\{1, 2, \ldots, 9\}\)
Sudoku: A Factor Graph for the Masses

rows are permutations of \{1, 2, \ldots, 9\}
columns are permutations of \{1, 2, \ldots, 9\}
subblocks are permutations of \{1, 2, \ldots, 9\}
Sudoku: A Factor Graph for the Masses

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- Rows are permutations of \{1, 2, \ldots, 9\}
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- Subblocks are permutations of \{1, 2, \ldots, 9\}

Implied factor graph has 81 variable and 27 factor nodes

\[
f(x) = \left( \prod_{i=1}^{9} f_{\sigma}(x_{i*}) \right) \left( \prod_{j=1}^{9} f_{\sigma}(x_{*j}) \right) \left( \prod_{k=1}^{9} f_{\sigma}(x_{B(k)}) \right) \prod_{(i,j) \in O} \mathbb{I}(x_{ij} = y_{ij})
\]
Consider any constraint satisfaction problem with observed entries:

- One can write $f(x)$ as the product of indicator functions.
- Some factors force $x$ to be valid (i.e., satisfy constraints).
- Other factors force $x$ to be compatible with observed values.
- Summing over $x$ counts the number of valid compatible sequences.
Consider any constraint satisfaction problem with observed entries.

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Low-complexity peeling solution:

- Set elements of $x$ one at a time.
- Each step looks for $i \in [n]$ and $x' \in \mathcal{X}$ such that:
  - For currently set variables, $f(x) = 0$ for all $x_i \in \mathcal{X} \setminus x'$
- Sudoku’s unique solution implies that $x_i = x'$ correct.
- Fix $x_i = x'$ and repeat until all values fixed.
One instance of 3-SAT is given, for example, by

\[ f(x) = (\overline{x}_1 \lor \overline{x}_3 \lor x_7) \land (x_1 \lor \overline{x}_2 \lor x_5) \land (x_2 \lor \overline{x}_4 \lor x_6). \]

In the FG, clause \( a \in [m] \) is enforced by the function \( f_a \).
Boolean Satisfiability: K-SAT

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- In the FG, clause \( a \in [m] \) is enforced by the function \( f_a \).

- Marginalization allows uniform sampling from valid set

  - For \( i = 1, 2, \ldots, n \), fix \( x_j \) for \( j < i \) and compute marginal

\[
g_i(x_i) = \frac{1}{Z_i} \sum_{x_{i+1}, \ldots, x_n} f(x) = \mathbb{P}(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1})
\]

  - Then, sample \( x_i \sim g_i(\cdot) \) and repeat
Boolean Satisfiability: K-SAT

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    \[ g_i(x_i) = \frac{1}{Z_i} \sum_{x_{i+1}, \ldots, x_n} f(x) = \mathbb{P}(X_i = x_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \]
  - Then, sample \( x_i \sim g_i(\cdot) \) and repeat

- This algorithm has low complexity if factor graph forms a tree
  - If not a tree, use approximate marginal from belief propagation
  - This is related to BP-guided decimation [MM09]
Low-Density Parity-Check (LDPC) Codes

- Linear codes defined by $xH^T = 0$ for all c.w. $x \in C \subset \{0, 1\}^n$
  - $H$ is an $m \times n$ sparse parity-check matrix for the code
  - Code bits and parity checks associated with cols/rows of $H$
Low-Density Parity-Check (LDPC) Codes

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  - Code bits and parity checks associated with cols/rows of $H$

- Factor graph: $H$ is the biadjacency matrix for variable/factor nodes
  - Ensemble defined by configuration model for random graphs
  - Checks define factors: $f_{\text{even}}(x_d^i) = \mathbb{I}(x_1 \oplus \cdots \oplus x_d = 0)$
  - Let $\mathbf{x}_{\partial a}$ be the subvector of variables in the $a$-th check and

$$f(x_1, \ldots, x_n) = \left( \prod_{a=1}^{m} f_{\text{even}}(\mathbf{x}_{\partial a}) \right) \left( \prod_{i=1}^{n} P_{Y|X}(y_i|x_i) \right)$$
A Little History

Robert Gallager introduced LDPC codes in 1962 paper

Low-Density Parity-Check Codes*

R. G. GALLAGER†

Summary—A low-density parity-check code is a code specified by a parity-check matrix with the following properties: each column contains a small fixed number $f > 3$ of 1's and each row contains a small fixed number $k > f$ of 1's. The typical minimum distance of these codes increases linearly with block length for a fixed rate and fixed $f$. When used with maximum likelihood decoding on a sufficiently quiet binary-input symmetric channel, the typical probability of decoding error decreases exponentially with block length for a fixed rate and fixed $f$.

A simple but nonoptimum decoding scheme operating directly from the channel a posteriori probabilities is described. Both the equations. We call the set of digits contained in a parity-check equation a parity-check set. For example, the first parity-check set in Fig. 1 is the set of digits $(1, 2, 3, 5)$.

The use of parity-check codes makes coding (as distinguished from decoding) relatively simple to implement. Also, as Elias [3] has shown, if a typical parity-check code of long block length is used on a binary symmetric channel, and if the code rate is between critical rate and channel capacity, then the probability of decoding error

Judea Pearl defined general belief-propagation in 1986 paper

Fusion, Propagation, and Structuring in Belief Networks*

Judea Pearl

Cognitive Systems Laboratory, Computer Science Department, University of California, Los Angeles, CA 90024, U.S.A.

Recommended by Patrick Hayes

ABSTRACT

Belief networks are directed acyclic graphs in which the nodes represent propositions (or variables), the arcs signify direct dependencies between the linked propositions, and the strengths of these dependencies are quantified by conditional probabilities. A network of this sort can be used to represent the generic knowledge of a domain expert, and it turns into a computational architecture if the links are used not merely for storing factual knowledge but also for directing and activating the data flow in the computations which manipulate this knowledge.
Simple Message-Passing Decoding for the BEC

- Constraint nodes define the valid patterns
  - Circles represent a single value shared by factors
  - Squares assert attached variables sum to 0 mod 2

- Iterative decoding on the binary erasure channel (BEC)
  - Messages passed in phases: bit-to-check and check-to-bit
  - Each output message depends on other input messages
  - Each message is either the correct value or an erasure
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Computation Graph and Density Evolution

\[ \tilde{x}_3 = \varepsilon y_2^3 \]
\[ y_2 = 1 - (1 - x_2)^3 \]
\[ x_2 = \varepsilon y_1^2 \]
\[ y_1 = 1 - (1 - x_1)^3 \]
\[ x_1 = \varepsilon \]

- Computation graph for a (3,4)-regular LDPC code
  - Illustrates decoding from the perspective of a single bit-node
  - For long random LDPC codes, the graph is typically a tree
  - Allows density evolution to track message erasure probability
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\begin{align*}
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\[
\tilde{x}_3 = \varepsilon y_2^3 \\
y_2 = 1 - (1 - x_2)^3 \\
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y_1 = 0.936 \\
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\[
\tilde{x}_3 = \varepsilon y_2^3
\]

\[
y_2 = 1 - (1 - x_2)^3
\]

\[
x_2 = 0.526
\]

\[
y_1 = 0.936
\]

\[
x_1 = 0.600
\]
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$$\tilde{x}_3 = \varepsilon y_2^3$$
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$$x_2 = 0.526$$
$$y_1 = 0.936$$
$$x_1 = 0.600$$

$$\varepsilon y^3$$

$$1 - (1 - x)^3$$
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\[ x_2 = 0.526 \]
\[ y_1 = 0.936 \]
\[ x_1 = 0.600 \]
Density Evolution (DE) for LDPC Codes

Density evolution for a (3, 4)-regular LDPC code:

\[ x_{\ell+1} = \varepsilon \left( 1 - (1 - x_\ell)^3 \right)^2 \]

Decoding Thresholds:

\[ \varepsilon^{BP} \approx 0.647 \]
\[ \varepsilon^{MAP} \approx 0.746 \]
\[ \varepsilon^{Sh} = 0.750 \]

- Binary erasure channel (BEC) with erasure prob. \( \varepsilon \)
- DE tracks bit-to-check msg erasure rate \( x_\ell \) after \( \ell \) iterations
- Defines noise threshold \( \varepsilon^{BP} \) for the large system limit
  - Easily computed numerically for given code ensemble
EXtrinsic Information Transfer (EXIT) Curves

- Introduced by ten Brink in 1999 to understand iterative decoding
- For the BEC, the MAP EXIT curve is

\[ h_{\text{MAP}}(\varepsilon) \triangleq \frac{1}{n} \sum_{i=1}^{n} H(X_i | Y_i \sim i(\varepsilon)) \]
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  - EXIT Area Theorem [ABK04]
    \[ \frac{1}{n} H(X | Y(\epsilon)) = \int_{0}^{\epsilon} h_{\text{MAP}}(\delta) d\delta \]
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- EXIT Area Theorem [ABK04]
  \[
  \frac{1}{n} H(X | \overline{Y}(\varepsilon)) = \int_{0}^{\varepsilon} h_{\text{MAP}}(\delta) d\delta
  \]

- BP EXIT curve
  \[
  h_{\text{BP}}(\varepsilon) \triangleq \frac{1}{n} \sum_{i=1}^{n} H(X_i | \Phi_i^{\text{BP}}(\overline{Y}_i(\varepsilon)))
  \]
  - where \( \Phi_i^{\text{BP}}(Z) \) is the BP estimate of \( X_i \) given \( Z \)
  - Data processing inequality: \( h_{\text{BP}}(\varepsilon) \geq h_{\text{MAP}}(\varepsilon) \)
EXtrinsic Information Transfer (EXIT) Curves

- (3,4)-regular LDPC code
- Codeword \((X_1, \ldots, X_n)\)
- Received \((Y_1, \ldots, Y_n)\)
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- BP EXIT curve via DE
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- 0 below BP threshold 0.647

\[ h_{BP}(\varepsilon) \]

\[ \varepsilon \]
EXtrinsic Information Transfer (EXIT) Curves

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  - 0 below MAP threshold 0.746
  - Area under curve equals rate \(R\)
  - Upper bounded by BP EXIT
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- MAP threshold upper bound \(\varepsilon_{MAP}\)
  - \(\varepsilon\) s.t. area under BP EXIT is \(R\)
Outline

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Abstract Formulation of Threshold Saturation
Properties of the MAP EXIT Curve

- For linear codes, the recovery of $X_i$ from $Y = y$
  - is independent of the transmitted codeword $X$
  - only depends on erasure indicator $z_i = 1(?) (y_i)$
  - is determined by whether $H(X_i | Z = z)$ is 0 or 1
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$$P_b(\varepsilon) = \mathbb{P}(Y_i = ?)H(X_i|\underline{Y}, Y_i = ?) = \varepsilon h^{\text{MAP}}(\varepsilon)$$
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- A sequence of rate-$R$ codes achieves capacity iff
  - $P_b(\varepsilon) \to 0$ for all $\varepsilon < 1 - R$
  - $h^\text{MAP}(\varepsilon) \to 0$ for all $\varepsilon < 1 - R$
  - $h^\text{MAP}(\varepsilon)$ transitions sharply from 0 to 1
For $\delta > 0$, transition width is $\varepsilon$-range over which $\delta \leq h^{\text{MAP}}(\varepsilon) \leq 1 - \delta$
The MAP EXIT Curve of a Capacity-Achieving Code

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EXIT Curves and Sharp Transitions

- Consider any monotone boolean function $f : \{0, 1\}^{n-1} \rightarrow \{0, 1\}$
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Define its symmetry group $\mathcal{G}$ to be

$$\mathcal{G} = \{ \pi \in S_{n-1} \mid f(\pi(z)) = f(z) \forall z \in \{0, 1\}^{n-1} \}$$
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Let $Z_i \in \{0, 1\}$ be i.i.d. with $P(Z_i = 1) = \varepsilon$ and define

$$h(\varepsilon) \triangleq \mathbb{E} [f(Z_1, \ldots, Z_{n-1})]$$

When do EXIT curves have a sharp transition? [KKMPSU15]

If the code’s permutation group is doubly transitive!

For example, Reed-Muller and prim. narrow-sense BCH codes

* Friedgut-Kalai’96: “Every monotone graph property has a sharp threshold”
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\]

If \( G \) is transitive, then \( h(\varepsilon) \) has transition width \( O\left(\frac{1}{\ln n}\right) \)
\[
\forall i, j \in \{1, 2, \ldots, n - 1\}, \exists \pi \in G \text{ s.t. } \pi(i) = j
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Summary and Open Problems

- Gallager's 1960 thesis already contains most of the tools necessary to achieve capacity in practice
  - But, he focuses mainly on the BSC
  - Had he attacked the BEC, practical capacity-achieving codes might have been introduced years earlier

- The first deterministic sequence of capacity-achieving binary codes for the BEC (under MAP decoding) was defined in 1954!
- Sequences of Reed-Muller codes achieve capacity on the BEC
- But, we didn't know this until 2015!

Open problems

- Generalize the Reed-Muller result to have weaker conditions and/or apply to more general channels/problems
- Find a purely information-theoretic proof of the Reed-Muller result for the BEC
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Outline

Introduction

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Message Passing

Applications of Factor Graphs

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Spatially-Coupled Factor Graphs

Universality for Multiuser Scenarios

Abstract Formulation of Threshold Saturation
What is Spatial Coupling?

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Variable nodes have a natural global orientation

Boundaries help variables to be recovered in an ordered fashion
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- Variable nodes have a natural global orientation
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Spatially-Coupled LDPC Codes: \((l, r, L, w)\) Ensemble

\[-L \quad \ldots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad \ldots \quad L\]
Spatially-Coupled LDPC Codes: \((l, r, L, w)\) Ensemble

- \(-L\) ... \(-2\) ... \(-1\) ... 0 ... 1 ... 2 ... \(L\)
- \(\pi_{-L}\) ... \(\pi_{-2}\) ... \(\pi_{-1}\) ... \(\pi_0\) ... \(\pi_1\) ... \(\pi_2\) ... \(\pi_L\)
- \(\pi'_{-L}\) ... \(\pi'_{-2}\) ... \(\pi'_{-1}\) ... \(\pi'_0\) ... \(\pi'_1\) ... \(\pi'_2\) ... \(\pi'_L\)

Historical Notes
- LDPC convolutional codes introduced by FZ in 1999
- Shown to have near optimal noise thresholds by LSZC in 2005
- \((l, r, L, w)\) ensemble proven to achieve capacity by KRU in 2011
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The LDPCC Gang

Iterative Decoding Threshold Analysis for LDPC Convolutional Codes

Michael Lentmaier, Member, IEEE, Arvind Sridharan, Member, IEEE, Daniel J. Costello, Jr., Life Fellow, IEEE, and Kamil Sh. Zigangirov, Fellow, IEEE
Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform So Well over the BEC

Shrinivas Kudekar, Member, IEEE, Thomas J. Richardson, Fellow, IEEE, and Rüdiger L. Urbanke
Density Evolution for the \((l, r, L, w)\)-SC LDPC Ensemble

\[(3, 4, 16, 3)\text{-SC Ensemble with } \varepsilon = 0.70\]

\[
x^{(\ell+1)}_i = \frac{1}{w} \sum_{k=0}^{w-1} \varepsilon \left( \frac{1}{w} \sum_{j=0}^{w-1} \left( 1 - (1 - x^{(\ell)}_{i+j-k})^{r-1} \right) \right)^{l-1} 1_{[-L, L+w-1]}(i-k)
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\((3, 4, 16, 3)\)-SC Ensemble with \(\varepsilon = 0.70\)

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\end{align*}
\]
Properties of Threshold Saturation

<table>
<thead>
<tr>
<th>$l$</th>
<th>$r$</th>
<th>$\varepsilon^{BP}$</th>
<th>$\varepsilon^{MAP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>0.4294</td>
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</tr>
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<tr>
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<td>0.4995</td>
</tr>
<tr>
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<td>0.3075</td>
<td>0.4999</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>0.2798</td>
<td>0.5000</td>
</tr>
</tbody>
</table>

- Spatial coupling achieves the MAP threshold as $w \to \infty$
  - BP threshold typically decreases after $l = 3$
  - MAP threshold is increasing in $l, r$ for fixed rate

Benefits and Drawbacks
- For fixed $L$, minimum distance grows linearly with block length
- Rate loss of $O(w/L)$ is a big obstacle in practice
Threshold Saturation via Spatial Coupling

- **General Phenomenon** (observed by Kudekar, Richardson, Urbanke)
  - BP threshold of the spatially-coupled system converges to the MAP threshold of the uncoupled system
  - Can be proven rigorously in many cases!
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- Connection to statistical physics
  - Factor graph defines system of coupled particles
  - Valid sequences are **ordered crystalline structures**

http://www.youtube.com/watch?v=Xe8vJrIvDQM
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- Connection to statistical physics
  - Factor graph defines system of coupled particles
  - Valid sequences are ordered crystalline structures

- Between BP and MAP threshold, system acts as supercooled liquid
  - Correct answer (crystalline state) has minimum energy
  - Crystallization (i.e., decoding) does not occur without a seed
  - Ex.: ice melts at 0°C but freezing w/o a seed requires −48.3°C

http://www.youtube.com/watch?v=Xe8vJrIvDQM
Why is Spatial Coupling Interesting?

- Breakthroughs: first practical constructions of
  - universal codes for binary-input memoryless channels [KRU12]
  - information-theoretically optimal compressive sensing [DJM11]
  - universal codes for Slepian-Wolf and MAC problems [YJNP11]
  - codes → capacity with iterative hard-decision decoding [JNP12]
  - codes → rate-distortion limit with iterative decoding [AMUV12]
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  - Our proof for increasing scalar/vector recursions [YJNP12/13]
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  - Our proof for increasing scalar/vector recursions [YJNP12/13]

- Spatial coupling as a proof technique [GMU13]
  - For a large random factor graph, construct a coupled version
  - Use DE to analyze BP decoding of coupled system
  - Compare uncoupled MAP with coupled BP via interpolation
Outline

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Message Passing

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Universality over Unknown Parameters

- The Achievable Channel Parameter Region (ACPR)
  - For a sequence of coding schemes involving one or more parameters, the parameter region where decoding succeeds in the limit
  - In contrast, a capacity region is a rate region for fixed channels

![Graph showing MAC-ACPR boundary for rate 1/2]
Universality over Unknown Parameters

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  - For a sequence of coding schemes involving one or more parameters, the parameter region where decoding succeeds in the limit
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- Properties
  - For fixed encoders, the ACPR depends on the decoders
  - For example, one has $\text{BP-ACPR} \subseteq \text{MAP-ACPR}$
  - Often, $\exists$ unique maximal ACPR given by information theory
Universality over Unknown Parameters

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- **Universality**
  - A sequence of encoding/decoding schemes is called *universal* if: its ACPR equals the optimal ACPR
  - Channel parameters are assumed unknown at the transmitter
  - At the receiver, the channel parameters are easily estimated
2-User Binary-Input Gaussian Multiple Access Channel

- Fixed noise variance
- Real channel gains $h_1$ and $h_2$ not known at transmitter
- Each code has rate $R$

MAC-ACPR denotes the information-theoretic optimal region
A Little History: SC for Multiple-Access (MAC) Channels

- KK consider a binary-adder erasure channel (ISIT 2011)
  - SC exhibits **threshold saturation** for the joint decoder

- YNPN consider the Gaussian MAC (ISIT/Allerton 2011)
  - SC exhibits **threshold saturation** for the joint decoder
  - For channel gains $h_1, h_2$ unknown at transmitter, SC provides **universality**

- Others consider CDMA systems without coding
  - TTK show SC improves BP demod of standard CDMA
  - ST prove saturation for a SC protograph-style CDMA
Spatially-Coupled Factor Graph for Joint Decoder
Spatially-Coupled Factor Graph for Joint Decoder

\[ 2L + 1 \]
Spatially-Coupled Factor Graph for Joint Decoder

$2L + 1$
MAC-ACPR boundary for rate $1/2$
DE Performance of the Joint Decoder

$\alpha_1 = |h_1|^2$

$\alpha_2 = |h_2|^2$

BP-ACPR, LDPC(3,6)

MAC-ACPR boundary for rate $1/2$
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BP-ACPR, LDPC\((3, 6)\)

BP-ACPR, SC\((3, 6, 64, 5)\)

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Introduction

Factor Graphs

Message Passing

Applications of Factor Graphs

Applications of EXIT Curves

Spatially-Coupled Factor Graphs

Universality for Multiuser Scenarios

Abstract Formulation of Threshold Saturation
Single Monotone Recursion

- Smooth increasing \( f : [0, 1] \rightarrow [0, 1] \)
Single Monotone Recursion

- Smooth increasing $f : [0, 1] \to [0, 1]$
- Discrete-time recursion
  \[ x^{(\ell+1)} = f(x^{(\ell)}) \]
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  \[ \frac{d}{dt} x(t) = f(x(t)) - x(t) = -\nabla U_s(x(t)) \]
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  \[ \frac{d}{dt} U_s(x(t)) = -(x(t) - f(x(t)))^2 \]
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Both $\downarrow 0$ iff no fixed points in $(0, 1]$
Coupled Monotone Recursion (1)

- Coupled recursion \( x^{(\ell+1)} = T x^{(\ell)} \) with \( x^{(\ell)} = (x_0^{(\ell)}, x_1^{(\ell)}, \ldots) \) and

\[
Tx \triangleq A^\top f(Ax),
\]

where \( [f(x)]_i = f(x_i) \) and \( A \) averages \( w \) adjacent values

\[
A = \frac{1}{w} \begin{bmatrix}
1 & 1 & \cdots & 1 & 0 & \cdots \\
0 & 1 & 1 & \cdots & 1 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

- i.e., avg right \( w \) positions, apply \( f \), then avg left \( w \) positions

\text{Danger: there be dragons ———– infinities}
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  $$A = \frac{1}{w} \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots \\ 0 & 1 & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \end{bmatrix}$$

  i.e., avg right $w$ positions, apply $f$, then avg left $w$ positions

- Coupled potential: $U_c(x) = \frac{1}{2} \sum_{i=0}^{\infty} x_i^2 - \sum_{i=0}^{\infty} F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_{i+j} \right)$

  Satisfies $\nabla U_c(x) = x - A^\top f(Ax)$

  Danger: there be dragons ———– infinities
Coupled Monotone Recursion (2)

- Properties of $T$ (note: $x \preceq y \iff x_i \leq y_i$ for all $i$)
  - $T$ is monotone: $x \preceq y$ implies $Tx \preceq Ty$
  - $T$ preserves spatial order: $x_{i+1} \geq x_i$ implies $[Tx]_{i+1} \geq [Tx]_i$
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For $x^{(0)} = 1$, iterates $x^{(\ell)}_i$ are decreasing in $\ell$ and increasing in $i$

- Spatial limit exists: $x^{(\ell)} = \lim_{i \to \infty} x^{(\ell)}_i$
- Iteration limit exists: $x^{(\infty)}_i = \lim_{\ell \to \infty} x^{(\ell)}_i$
- Iteration limit satisfies fixed point: $x^{(\infty)} = Tx^{(\infty)}$
- Double limit satisfies fixed point: $x^{(\infty)} = f(x^{(\infty)})$
Intuition Behind Threshold Saturation

- Between the BP and MAP threshold
  - decoding trajectory looks like a right-moving wave

\[ S_x(i) = x_i - 1 \] with \( x_1 = 0 \)

Relative potential:
\[
V_x(t) = U_c((1-t)x + tS_x) - U_c(x)
\]

If \( x_{i+1} \geq x_i \) for all \( i \), then \( V_x(t) \) well-defined for \( t \in [0, 1] \)

For \( t = 1 \), one gets a telescoping sum that shows \( V_x(1) \leq -U_s(x_\infty) \)
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- Right-shift $S$ satisfies $[Sx]_i = x_{i-1}$ with $x_{-1} = 0$
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  \[ V_x(1) \leq -U_s(x_\infty) \]
Threshold Saturation

**Theorem**

If \( f(0) = 0 \) and \( f'(0) < 1 \) (0 is stable f.p.) with \( U_s(x) > 0 \) for \( x \in (0, 1] \), then there exists \( w_0 < \infty \) such that \( x^{(\infty)} = 0 \) for all \( w > w_0 \).
Theorem

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Define relative potential (with \( x_i(t) \triangleq x_i + t(x_{i-1} - x_i) \))

\[
V_{\overline{x}}(t) \triangleq \frac{1}{2} \sum_{i=0}^{\infty} (x_i(t)^2 - (x_i)^2) - \sum_{i=0}^{\infty} \left[ F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i(t) \right) - F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i \right) \right]
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Sketch of Proof:
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- Sketch of Proof:
  - For \( x^{(0)} = 1 \), let \( z = x^{(\infty)} \) be limiting fixed-point of recursion
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If $f(0) = 0$ and $f'(0) < 1$ ($0$ is stable f.p.) with $U_s(x) > 0$ for $x \in (0, 1]$, then $\exists w_0 < \infty$ such that $x^{(\infty)} = 0$ for all $w > w_0$.

Define relative potential (with $x_i(t) \triangleq x_i + t(x_{i-1} - x_i)$)

$$V_x(t) \triangleq \frac{1}{2} \sum_{i=0}^{\infty} (x_i(t)^2 - (x_i)^2) - \sum_{i=0}^{\infty} \left[ F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i(t) \right) - F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i \right) \right]$$

Sketch of Proof:

- For $x^{(0)} = 1$, let $z = x^{(\infty)}$ be limiting fixed-point of recursion
- If $z_{\infty} = 0$, then we’re done. Suppose $z_{\infty} > 0$
**Threshold Saturation**

**Theorem**

If \( f(0) = 0 \) and \( f'(0) < 1 \) (0 is stable f.p.) with \( U_s(x) > 0 \) for \( x \in (0, 1] \), then \( \exists w_0 < \infty \) such that \( x^{(\infty)} = 0 \) for all \( w > w_0 \).

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- Sketch of Proof:
  - For \( x^{(0)} = 1 \), let \( z = x^{(\infty)} \) be limiting fixed-point of recursion
  - If \( z_{\infty} = 0 \), then we’re done. Suppose \( z_{\infty} > 0 \)
  - Then, \( z_{\infty} = f(z_{\infty}) \geq \) smallest non-zero f.p. > 0 (ind. of \( w \))
**Threshold Saturation**

**Theorem**

If $f(0) = 0$ and $f'(0) < 1$ (0 is stable f.p.) with $U_s(x) > 0$ for $x \in (0, 1]$, then $\exists w_0 < \infty$ such that $x^{(\infty)} = 0$ for all $w > w_0$.

- Define **relative potential** (with $x_i(t) \triangleq x_i + t(x_{i-1} - x_i)$)

$$V_x(t) \triangleq \frac{1}{2} \sum_{i=0}^{\infty} (x_i(t)^2 - (x_i)^2) - \sum_{i=0}^{\infty} \left[ F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i(t) \right) - F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i \right) \right]$$

- **Sketch of Proof:**
  - For $x^{(0)} = \underline{1}$, let $z = x^{(\infty)}$ be limiting fixed-point of recursion
  - If $z_{\infty} = 0$, then we’re done. **Suppose** $z_{\infty} > 0$
  - Then, $z_{\infty} = f(z_{\infty}) \geq$ smallest non-zero f.p. $> 0$ (ind. of $w$)
  - Thus, $U(z_{\infty}) > 0$ by hypothesis
Threshold Saturation

**Theorem**

If $f(0) = 0$ and $f'(0) < 1$ ($0$ is stable f.p.) with $U_s(x) > 0$ for $x \in (0, 1]$, then $\exists w_0 < \infty$ such that $x_{\infty} = 0$ for all $w > w_0$.

- Define relative potential (with $x_i(t) \triangleq x_i + t(x_{i-1} - x_i)$)

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- Sketch of Proof:
  - For $x^{(0)} = 1$, let $z = x^{(\infty)}$ be limiting fixed-point of recursion
  - If $z_\infty = 0$, then we’re done. Suppose $z_\infty > 0$
  - Then, $z_\infty = f(z_\infty) \geq$ smallest non-zero f.p. $> 0$ (ind. of $w$)
  - Thus, $U(z_\infty) > 0$ by hypothesis
  - Telescoping sum for $V$ shows $V_z(1) \leq -U(z_\infty) < 0$
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- Define relative potential (with \( x_i(t) \equiv x_i + t(x_{i-1} - x_i) \))

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V_x(t) \equiv \frac{1}{2} \sum_{i=0}^{\infty} (x_i(t)^2 - (x_i)^2) - \sum_{i=0}^{\infty} \left[ F\left( \frac{1}{w} \sum_{j=0}^{w-1} x_i(t) \right) - F\left( \frac{1}{w} \sum_{j=0}^{w-1} x_i \right) \right]
\]

- Sketch of Proof:
  - For \( \bar{x}^{(0)} = 1 \), let \( \bar{z} = x^{(\infty)} \) be limiting fixed-point of recursion
  - If \( z_\infty = 0 \), then we’re done. Suppose \( z_\infty > 0 \)
  - Then, \( z_\infty = f(z_\infty) \geq \) smallest non-zero f.p. > 0 (ind. of \( w \))
  - Thus, \( U(z_\infty) > 0 \) by hypothesis
  - Telescoping sum for \( V \) shows \( V_{\bar{z}}(1) \leq -U(z_\infty) < 0 \)
  - Taylor series for \( V \) shows \( |V_{\bar{z}}(1)| \leq K \frac{1}{w} (1 + \sup_{x \in [0,1]} |f'(x)|) \)
Threshold Saturation

**Theorem**

If \( f(0) = 0 \) and \( f'(0) < 1 \) (0 is stable f.p.) with \( U_s(x) > 0 \) for \( x \in (0, 1] \), then \( \exists w_0 < \infty \) such that \( x_{\infty} = 0 \) for all \( w > w_0 \).

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V_{\bar{x}}(t) \triangleq \frac{1}{2} \sum_{i=0}^{\infty} \left( x_i(t)^2 - (x_i)^2 \right) - \sum_{i=0}^{\infty} \left[ F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i(t) \right) - F \left( \frac{1}{w} \sum_{j=0}^{w-1} x_i \right) \right]
\]

- Sketch of Proof:
  - For \( \bar{x}^{(0)} = \mathbf{1} \), let \( \bar{z} = \bar{x}^{(\infty)} \) be limiting fixed-point of recursion
  - If \( \bar{z}_\infty = 0 \), then we’re done. **Suppose \( \bar{z}_\infty > 0 \)**
  - Then, \( \bar{z}_\infty = f(\bar{z}_\infty) \geq \) smallest non-zero f.p. > 0 (ind. of \( w \))
  - Thus, \( U(\bar{z}_\infty) > 0 \) by hypothesis
  - Telescoping sum for \( V \) shows \( V_{\bar{z}}(1) \leq -U(\bar{z}_\infty) < 0 \)
  - Taylor series for \( V \) shows \( |V_{\bar{z}}(1)| \leq K \frac{1}{w} \left( 1 + \sup_{x \in [0,1]} |f'(x)| \right) \)
  - Thus, we get a **contradiction for sufficiently large \( w \)**
History of Threshold Saturation Proofs

- the BEC in 2010 [KRU11]
  - Established many properties and tools used by later approaches
- the Curie-Weiss model of physics in 2010 [HMU12]
- CDMA using a GA in 2011 [TTK12]
- CDMA with outer code via GA in 2011 [Tru12]
- compressive sensing using a GA in 2011 [DJM13]
- regular codes on BMS channels in 2012 [KRU13]
- increasing scalar and vector recursions in 2012 [YJNP14]
- irregular LDPC codes on BMS channels in 2012 [KYMP14]
- non-decreasing scalar recursions in 2012 [KRU15]
- non-binary LDPC codes on the BEC in 2014 [AG16]
- and more since 2014...
Summary and Open Problems

- **Factor Graphs**
  - **Useful tool** for modeling dependent random variables
  - Low-complexity algorithms for approximate inference
  - Density evolution can be used to analyze performance

- **Spatial Coupling**
  - **Powerful technique** for designing and understanding FGs.
  - Related to the statistical physics of supercooled liquids
  - **Simple proof** of threshold saturation for scalar recursions

- **Interesting Open Problems**
  - Code constructions that reduce the rate-loss due to termination
  - Compute the scaling exponent for SC codes
  - Finding new problems where SC provides benefits
Thanks for your attention
References I


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