Network Information Theory and the Entropy Power Inequality

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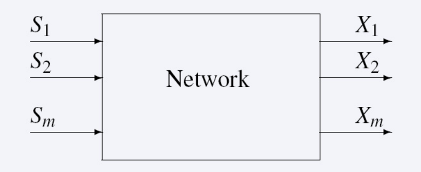
Motivation

- Growing interest in information transmission over networks and hence *multi-user information theory*
    - wired networks, e.g. the Internet
    - emerging wireless networks, e.g. sensor networks, ad hoc networks, etc.
- Many multi-user information theory problems still open (in contrast to the point-to-point case)
  $\Rightarrow$ Much less deployment of information theory in the design of current networks
- Difficulties in developing *network information theory*
- One of the fundamental difficulties: *Characterizing the Entropy Region*

A Network Problem

Consider an acyclic discrete memory-less network where transmitter S_i needs to communicate with the receiver X_i at rate R_i



The *rate region* for reliable communication is (cf. van der Meulen, Ahlswede)

$$\mathcal{R} = \text{cl}\left\{ R_i, i = 1, \ldots, m \;\middle|\; R_i < \frac{1}{T}\left(H(X_i^T) - H(X_i^T | S_i^T) \right) \right\} \quad \text{as } T \to \infty$$

- ${S_i}^T$, ${X_i}^T$: concatenated over T channel uses
- While not a surprising characterization, it is difficult to compute

- Equivalently we can represent the rate region through its tangent hyperplanes, for which we need to solve,

$$\lim_{T\to\infty} \sup_{p(S_i^T) \text{ and network operations}} \sum_{i=1}^{m} \alpha_i \frac{1}{T} \left(H(X_i^T) - H(X_i^T|S_i^T) \right)$$

- An extremely difficult problem, since
  - it is infinite-dimensional (*infinite-letter characterization*)
  - for any T, the problem is highly non-convex in the $p(S_i^T)$ and the "network operations"

Can these issues be resolved?

Entropy Vectors

- Notion of **Entropy Vectors** simplifies such problem
- **Definition:**
  - Consider n discrete random variables with alphabet-size N. For any set $\alpha \subseteq \{1, \ldots, n\}$, the *normalized entropy* is

    $$h_\alpha = \frac{1}{\log N} H(X_i, i \in \alpha).$$

    The $2^n - 1$ dimensional vector obtained from these entropies is called a *normalized entropy vector*.
  - Conversely, any $2^n - 1$ dimensional vector which can be regarded as the normalized entropy vector of some collection of n random variables, for some alphabet size N, is called *normalized-entropic*.
- Similarly we can define non-normalized entropy vectors through

$$H_\alpha = H(X_i, i \in \alpha)$$

- Γ_n^*: The space of all non-normalized entropic vectors
- Ω_n^*: The region of all normalized entropy vectors
- Focusing on *normalized* entropy seems to be more natural
  - it comes up in

$$\sum_{i=1}^{m} \alpha_i \frac{1}{T} \left(H(X_i^T) + H(S_i^T) - H(X_i^T, S_i^T) \right)$$

  - It makes the entropy region finite,

$$h_\alpha \leq |\alpha|$$

Theorem (Ω_n^* and Γ_n^*)

Let $\mathrm{cone}(S) = \{\alpha X | \alpha \geq 0, X \in S\}$*. Then we have,*

$$\mathrm{cone}(\overline{\Omega}_n^*) = \overline{\Gamma}_n^*$$

Moreover $\overline{\Omega}_n^*$ *and* $\overline{\Gamma}_n^*$ *are convex sets.*

Network Problem Revisited

Thus the network problem can be rewritten as

$$\sup \alpha^T h,$$

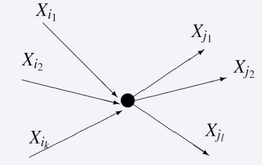
subject to $h \in \Omega_n^*$ (where n is the number of random variables in the network) and subject to the network constraints:

1. topological constraints
2. channel constraints

Let's examine these constraints...

Topological Constraints

- Let i be a non-source node, receiving the signals $X_{i_1}, \ldots, X_{i_k}$ and transmitting the signals $X_{j_1}, \ldots, X_{j_l}$



The following linear constraint is the result of causality:

$$h(X_{j_q}, X_{i_1}, \ldots, X_{i_k}) - h(X_{i_1}, \ldots, X_{i_k}) = 0 \quad q = 1, \ldots l$$

- At source nodes,
  $h(S_i, S_j) - h(S_i) - h(S_j) = 0$ if S_i and S_j are independent and
  $h(S_i, S_j) = h(S_i) = h(S_j)$ if $S_i = S_j$

Channel Constraints

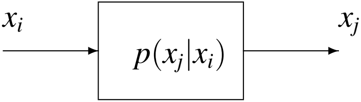
- Channels constrain the joint distribution of all random variables in the network $\Longrightarrow$ admissible entropy vectors:

  $$p(X_i, X_j) = p(X_j|X_i)p(X_i),$$

  or, equivalently,

  $$\int \prod_{k \neq i,j} dX_k \, p(X_1, \ldots, X_n) = p(X_j|X_i) \int \prod_{k \neq i} dX_k \, p(X_1, \ldots, X_n),$$

  which is a linear constraint on the joint distribution



- Channel constraints do not translate directly to entropies
- Let $\Omega^*_{n,C}$ denote the space of entropic vectors constrained by the discrete memoryless channels in the network

Back to the Network Problem

Characterizing $\Omega^*_{n,C}$ is not an easy task in general, however the region of channel constrained entropy vectors remains convex,

Theorem (Network Problem as a Convex Optimization)

Network problem can be cast as a convex optimization over the set of channel constrained entropy vectors:

$$\max_{h \in \Omega^*_{n,C}, Ah=0} \alpha^T h,$$

where $Ah = 0$ *represents the topological constraints.*

- Circumvented both the *infinite-letter characterization* problem, as well as the *non-convexity*
- *Characterization of* $\Omega^*_{n,C}$ *is of fundamental importance in network information theory*

Some Consequences: Duality and Cutset bounds

Convex optimization machinery, yields interesting consequences, e.g. using the duality argument we obtain:

$$\max_{h\in\Omega^*_{n,C}, Ah=0} \alpha^T h = \max_{h\in\Omega^*_{n,C}} \min_{\lambda} \left(\alpha^T h + \lambda^T A h\right) = \min_{\lambda} \max_{h\in\Omega^*_{n,C}} (\alpha + A^T\lambda)^T h$$

Now any choice of λ yields an upper bound:

$$\max_{h\in\Omega^*_{n,C}, Ah=0} \alpha^T h \leq \max_{h\in\Omega^*_{n,C}} (\alpha + A^T\lambda)^T h$$

- Partition the nodes of the network into two sets, one containing sources and the other the destinations,
- If i is a node whose edges do not cross the cut, set $\lambda_i = 0$
- Optimizing over the remaining entries of λ boils down to the point-to-point problem

Wired vs. Wireless

- Characterizing $\Omega^*_{n,C}$ is formidable in general
- For wired networks things simplify,

Theorem

The network problem for wired networks can be cast as a convex optimization over the unconstrained *entropy region,*

$$\max_{h \in \Omega^*_n, h_{1:n} \le c, Ah = 0} \alpha^T h.$$

- For wireless networks, interaction of signals such as interference should be considered
  $\Longrightarrow$ Studying bounds on the entropy of sums of independent random variables

Entropy Region of $x,\ y$ and $z = x + y$

Let $x,\ y \in \mathcal{R}^m$ be independent vector-valued continuous random variables, $z = x + y$ and h the corresponding *normalized* entropy vector

Define,

$$\Psi_3 : \text{the entropy region of } x,\ y \text{ and } z$$

$$\Upsilon = \{h | h_{xy} = h_{xz} = h_{yz} = h_x + h_y,\ h_{xyz} = -\infty\},$$

$$\Xi = \{h | h_z \geq \frac{1}{2}\log(e^{2h_x} + e^{2h_y})\}.$$

Note that the set Ξ contains the points which satisfy the so-called *Entropy Power Inequality*

Theorem (Entropy region of $x,\ y$ and $z = x + y$)

$$\Psi_3 = \Gamma_3 \cap \Upsilon \cap \Xi.$$

Proof:

- from assumptions and the classical EPI, $\Psi_3 \subseteq \Gamma_3 \cap \Upsilon \cap \Xi$
- lower bound for EPI is achieved by Gaussian variables with proportional covariance matrices
- for fixed h_x and h_y, h_z can grow unbounded,

$$x : \mathcal{N}(0, \epsilon I_m + \sigma_x^2 U_x U_x^T)$$
$$y : \mathcal{N}(0, \epsilon I_m + \sigma_y^2 U_y U_y^T)$$

where U_x, U_y are $m \times m/2$ unitary matrices s.t,

$$U_x^T U_x = U_y^T U_y = I_{m/2} \text{ and } U_x^T U_y = 0$$

Calculating normalized entropy of Gaussians give,

$$h_x = \frac{1}{2}\log 2\pi e + \frac{1}{4}\log \epsilon(\epsilon + \sigma_x^2),$$
$$h_y = \frac{1}{2}\log 2\pi e + \frac{1}{4}\log \epsilon(\epsilon + \sigma_y^2).$$

since,

$$z : \mathcal{N}(0, 2\epsilon I_m + \sigma_x^2 U_x U_x^T + \sigma_y^2 U_y U_y^T),$$

entropy of z can be computed in the same fashion

$$h_z = \frac{1}{2}\log 2\pi e + \frac{1}{4}\log(2\epsilon + \sigma_x^2)(2\epsilon + \sigma_y^2).$$

Let,

$$\sigma_x^2 = \frac{e^{4c_x}}{(2\pi e)^2 \epsilon} - \epsilon \quad , \quad \sigma_y^2 = \frac{e^{4c_y}}{(2\pi e)^2 \epsilon} - \epsilon$$

for some finite positive c_x and c_y and let $\epsilon \to 0$, therefore we obtain,

$$h_x = c_x, \quad h_y = c_y, \quad h_z \to \infty.$$

- noting that Ξ is a convex set, completes the proof.

Conclusions

- Large class of network information theory problems can be cast as convex optimization problems over the convex set of *channel-constrained entropies*
- Cut-set bounds follow from convex optimization duality
- Characterizing Γ_n^* is of fundamental importance; open for $n \geq 4$
- Identifying the entropy region of sums of independent variables is important for wireless networks
- This region was obtained for 3 independent vector-valued continuously distributed random variables, $x,\ y$ and $z = x + y$; Entropy power inequality turns out to be the main constraining inequality in this case