

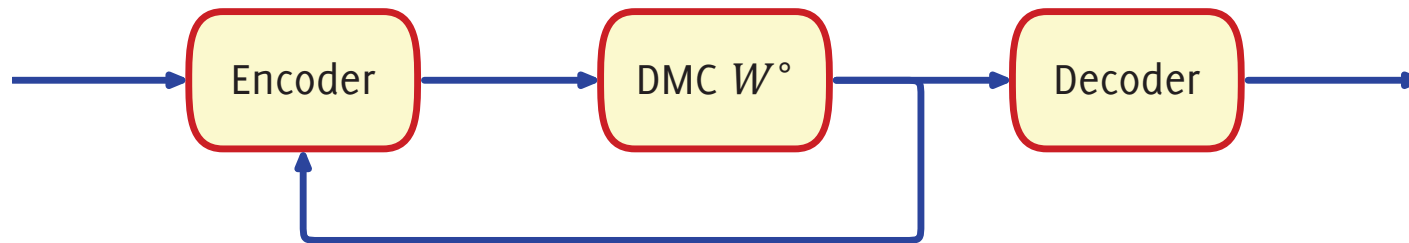
# Communicating over unknown channels

How good are training based schemes?

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School of Information Theory, Aug 12, 2009

# PROBLEM FORMULATION



## ■ Model

- $W^\circ$  belongs to a family  $\mathcal{W}$ .
- The family  $\mathcal{W}$  is known, but  $W^\circ$  is not.

## ■ Capacity: $C(W^\circ)$ can be achieved

Error exponent of universal schemes at rate  $R < C(W^\circ)$ .

# LITERATURE OVERVIEW

## ■ When $W^\circ$ is known

- Burnashev Error Exponent  $E_B(R, W^\circ) = D(1 - R/C(W^\circ))$ .
- Simple achievability scheme due to Yamamoto and Itoh

## ■ When $W^\circ$ is unknown

- Burnashev Exponent cannot be achieved universally  
*(Tchamkerten and Telatar)*
- Error exponent of **simple training based schemes** doesn't have positive slope at capacity.  
*(Tchamkerten and Telatar)*
- When feedback is absent, training based schemes cannot achieve universal decoding exponent  
*(Feder and Lapidoth)*

# INTUITION FOR OBTAINING GOOD TRAINING BASED SCHEMES

- Use Yamamoto-Itoh like scheme
  - Simple two phase scheme: communication phase and control phase.
  - The exponent of the communication phase only needs to be positive.
- Train more than once (always retrain from scratch)
  - Ensures independence of events
  - The training exponent does not dominate

# A MOTIVATING EXAMPLE

## ■ Model

$$\mathcal{W}_p = \{\text{BSC}(p), \text{BSC}(1 - p)\}, \quad p \in [0, 1/2)$$

■ Need to communicate at rate  $R < C_p = 1 - H(p)$ .

■ The Coding scheme: Multiple epochs of length  $t$

- Training phase of length  $\beta_1 t$ : Send  $\beta_1 t$  zeros.
- Communication phase of length  $\beta_2 t$ : Send  $\lfloor 2^{tR} \rfloor$  messages at rate  $R/\beta_2$ .
- Re-training phase of length  $\beta_3 t$ : Send  $\beta_3 t$  zeros.
- Confirmation phase of length  $\beta_4 t$ : Send  $\beta_4 t$  zeros (ACK) or  $\beta_4 t$  ones (NACK).

# PERFORMANCE

For epoch of length  $t$ , let

- $M(t) = 2^{tR}$  (number of messages)
- $P_e(t)$  = probability of error
- $T(t)$  = (random) length of communication

## ■ Rate

$$\lim_{t \rightarrow \infty} \frac{\log M(t)}{\mathbf{E}\{T(t)\}} = R.$$

## ■ Error Exponent

$$E_s = - \lim_{t \rightarrow \infty} \frac{\log P_e(t)}{\mathbf{E}\{T(t)\}} \geq \underbrace{\alpha D(p \| 1 - p)}_{\text{Burnashev's Exp}} \left( 1 - \frac{R}{C_p} \right)$$

where  $\alpha = \frac{D(0.5 \| p)}{D(0.5 \| p) + D(p \| 1 - p)}$

and  $D(x \| y) = x \log(x/y) + (1 - x) \log((1 - x)/(1 - y))$ .

# CONCLUSION

- The training based scheme is simple.
- Comes within a constant fraction of the Burnashev's exponent.
- For this example, Tchamkerten and Telatar have shown that no scheme can achieve Burnashev exponent.
- **Future Directions**
  - More general family of channels
  - Rate adaptation
- **Open Questions**
  - Error exponent of best training based scheme
  - Error exponent of universal schemes

## ANALYSIS: NOTATION

For epoch length  $t$  and epoch  $k$ ,  $k = 1, 2, \dots$  define the events

- $E_1(t, k)$ : channel estimation at end of the first phase is incorrect.
- $D(t, k)$ : decoding at the end of second phase is incorrect.
- $E_2(t, k)$ : channel estimation at end of third phase is incorrect.
- $C_A(t, k)$ : ACK is decoded as NACK in the fourth phase.
- $C_N(t, k)$ : NACK is decoded as ACK in the fourth phase.

## ANALYSIS: ASSUMPTIONS

- Codebook of phase two has error exponent  $E_R$ . Thus, for all  $k$ ,

$$-\lim_{t \rightarrow \infty} \frac{\Pr\{D(t, k) \mid \bar{E}_1(t, k)\}}{\beta_2 t} = E_R.$$

When  $R/\beta_2 < C_p$ , we have  $E_R > 0$  and  $\lim_{t \rightarrow \infty} \Pr\{D(t, k) \mid \bar{E}_1(t, k)\} = 0$ .

- The decoding regions for phase four are chosen such that

$$\lim_{t \rightarrow \infty} \Pr\{C_A(t, k) \mid \bar{E}_2(t, k)\} = 0,$$

$$\lim_{t \rightarrow \infty} \Pr\{C_B(t, k) \mid \bar{E}_2(t, k)\} = 0,$$

and

$$\lim_{t \rightarrow \infty} \frac{\Pr\{C_A(t, k) \mid \bar{E}_2(t, k)\}}{\beta_4 t} = D(p \parallel 1 - p).$$

## ANALYSIS: A LEMMA

Let  $\zeta(t)$  be the epoch of the stopping time when communication stops. That is,

$$\zeta(t) = \inf\{k \in \mathbb{N} : D(t,k) \cap C_N(t,k) \cup \bar{D}(t,k) \cap \bar{C}_A(t,k)\}$$

Then, for all  $t$ ,  $T(t) = t\zeta(t)$  almost surely, and

$$\lim_{t \rightarrow \infty} \mathbf{E}\{\zeta(t)\} = 1.$$

**Proof Outline:**  $\zeta(t)$  is stochastically dominated by a random variable with Geometric  $\xi(t)$  distribution, where

$$\zeta(t) = \Pr\{D(t,1) \mid \bar{E}_1(t,1)\} + \Pr\{C_N(t,1) \mid \bar{E}_2(t,1)\}.$$

Thus,  $\Pr\{\zeta(t) \geq k\} \leq \xi(t)^{k-1}$  which implies  $\mathbf{E}\{\zeta(t)\} \leq 1/(1 - \xi(t))$ . The result follows from the fact that  $\lim_{t \rightarrow \infty} \xi(t) = 0$ .

# ANALYSIS: MAIN STEPS OF THE PROOF

## ■ Rate of communication

$$\lim_{t \rightarrow \infty} \frac{\log M(t)}{\mathbf{E}\{T(t)\}} = \frac{\log 2^t R}{\mathbf{E}\{T(t)\}} = R \lim_{t \rightarrow \infty} \frac{1}{\mathbf{E}\{\zeta(t)\}} = R$$

## ■ Probability of error

$$P_e(t) \leq \Pr\{D(t,1)\} \Pr\{C_N(t,1)\} \frac{1}{1 - \xi(t)}$$

## ■ Error exponent

$$E_s = - \lim_{t \rightarrow \infty} \frac{\log P_e(t)}{\mathbf{E}\{T(t)\}} \geq - \lim_{t \rightarrow \infty} \frac{\log \Pr\{D(t,1)\}}{t} - \lim_{t \rightarrow \infty} \frac{\log \Pr\{C_N(t,1)\}}{t}$$

# ANALYSIS: MAIN STEPS OF THE PROOF

## ■ The first term

$$\begin{aligned}\Pr\{D(t, 1)\} &\leq \Pr\{E_1(t, 1)\} + \Pr\{D(t, 1) \mid \bar{E}_1(t, 1)\} \\ &\leq \exp(-\beta_1 t D(0.5\|p)) + \exp(-\beta_2 t E_R(R/\beta_2))\end{aligned}$$

## ■ The second term

$$\begin{aligned}\Pr\{C_N(t, 1)\} &\leq \Pr\{E_2(t, 1)\} + \Pr\{C_N(t, 1) \mid \bar{E}_2(t, 1)\} \\ &\leq \exp(-\beta_3 t D(0.5\|p)) + \exp(-\beta_4 t D(p\|1 - p))\end{aligned}$$

## ■ Combining,

$$\begin{aligned}-\lim_{t \rightarrow \infty} \frac{\log \Pr\{D(t, 1)\}}{t} &\geq \lim_{t \rightarrow \infty} \min\{\beta_1 D(0.5\|p), \beta_2 E_R(R/\beta_2)\} \geq 0 \\ -\lim_{t \rightarrow \infty} \frac{\log \Pr\{C_N(t, 1)\}}{t} &\geq \lim_{t \rightarrow \infty} \min\{\beta_3 t D(0.5\|p), \beta_4 t D(p\|1 - p)\}\end{aligned}$$

## ANALYSIS: MAIN STEPS OF THE PROOF

- Take  $\beta_3/\beta_4 = D(p\|1-p)/D(0.5\|p)$ . Then,

$$-\lim_{t \rightarrow \infty} \frac{\log \Pr\{C_N(t, 1)\}}{t} \geq \frac{D(0.5\|p)D(p\|1-p)}{D(0.5\|p) + D(p\|1-p)} \lim_{t \rightarrow \infty} (\beta_3 + \beta_4)$$

- Choose  $\beta_1 = o(1)$  and  $\beta_2 = R/C_p - o(1)$ .

- $\beta_3 + \beta_4 = 1 - \beta_1 - \beta_2 = 1 - R/C_p + o(1)$ .

- Combining, we have

$$E_S \geq \left(1 - \frac{R}{C_p}\right) \frac{D(0.5\|p)D(p\|1-p)}{D(0.5\|p) + D(p\|1-p)}$$