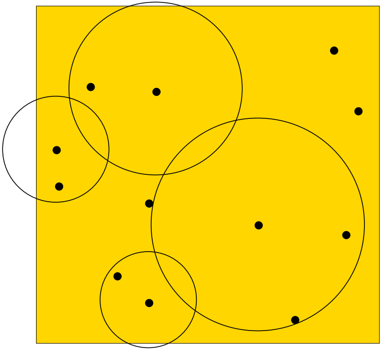
The Transport Capacity of a Wireless Network is a Subadditive Euclidean Functional

Radha Krishna Ganti and Martin Haenggi

August 5, 2009

Consider n points $x_1, \ldots, x_n$ distributed uniformly on the unit square $[0, 1]^2$.



- Length of a minimum spanning tree: $\frac{L_s(m)}{\sqrt{n}} \to C_1$
- Length of a minimal matching: $\frac{L_m(m)}{\sqrt{n}} \to C_2$ item <1-> Length of a cycle which satisfies the Euclidean TSP: $\frac{L_T(m)}{\sqrt{n}} \to C_3$
- Transport capacity of these *wireless nodes* is

$$T(n) = \Theta(\sqrt{n})$$

All these quantities are **Subadditive Euclidean Functionals.**

Subadditive Euclidean Functional

Let T be a real valued function of the finite subsets of $\mathbb{R}^2$ and $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^2$

Euclidean functional:

1. $T(aX) = aT(X)$ for all $a > 0$
2. $T(X + x) = T(X)$ for all $x \in \mathbb{R}^2$

Monotone property:

1. $T(\{x\} \cup X) \geq T(X)$ for any $x \in \mathbb{R}^2$.

Subaddititvity

Let Q_i be a partition of the $[0,1]^2$ into sub-squares of side m^{-1}. T is subadditve if for all $t > 0$ there exists a constant C not depending on t and m such that

$$T(X \cap [0,t]^2) \leq \sum_{i=1}^{m^2} T(X \cap tQ_i) + Ctm$$

Theorem (Steele (1981))

Suppose T is a Euclidean functional on $\mathbb{R}^2$ which is monotone and subadditive. If $\{x_i : 1 \leq i < \infty\}$ are independent and uniformly distributed on $[0,1]^2$, then there is a constant $C(T)$ such that

$$\lim_{n\to\infty} \frac{T\Big(x_1,\ldots,x_n\Big)}{\sqrt{n}} = C(T)$$

with probability one.

Transport Capacity (TC)

Let $x_1, \ldots, x_n$ be n nodes uniformly distributed in the unit square $[0,1]^2$.

Protocol Model

A transmitter x_i can connect to receiver x_j denoted by $(x_i \to x_j)$ if the disk $B(x_j, \beta\|x_i - x_j\|), \beta > 1$ does not contain any other transmitters.

Transport Capacity

The transport capacity of these n nodes is defined as

$$T\left(\{x_1, x_2, \cdots, x_n\}\right) = \sup_{\mathcal{S}} \left[\sum_{(i,j) \in \{1,2..n\}^2} \lambda_{ij} \|x_i - x_j\| \right]$$

where the supremum is taken over the supportable rate pairs $\mathcal{S}$ where the nodes communicate using the protocol model.

Gupta,Kumar (2000) and Franceschetti (2007)

When interference is treated as noise,

$$C_1\sqrt{n} \leq T(X_n) \leq C_2\sqrt{n}$$

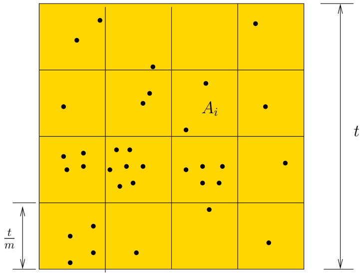
where $X_n = \{x_1, x_2, \cdots, x_n\}$.

We want to prove

$$\lim_{n\to\infty} \frac{T(X_n)}{\sqrt{n}} = A_2$$

with probability one.

Subadditivity of TC



Theorem (Cutting Lemma, Ganti and Haenggi (2008))

Consider a square $A = [0, t]^2 \subset \mathbb{R}^2$ and let $X = \{x_1 \ldots x_k\} \subset A$ denote a set of k nodes. Divide A into m^2 squares of equal sides with length t/m and denote each square by A_i. We then have

$$T(X) \leq \sum_{i=1}^{m^2} T(X \cap A_i) + Cmt$$

TC is a subadditive Euclidean functional

Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^2$

1. $T(aX) = aT(X)$ for all $a > 0$
2. $T(X + x) = T(X)$ for all $x \in \mathbb{R}^2$
3. Monotone property: $T(\{x\} \cup X) \geq T(X)$ for any $x \in \mathbb{R}^2$.
4. Let Q_i be a partition of the $[0,1]^2$ into sub-squares of side m^{-1}

$$T(X \cap [0,t]^2) \leq \sum_{i=1}^{m^2} T(X \cap tQ_i) + Ctm$$

Hence by Steele's theorem we have

$$\lim_{n\to\infty} \frac{T(X_n)}{\sqrt{n}} = A_2$$

with probability one.

Concentration around the mean

By a Theorem of Rhee on subadditive functionals we also have

$$\mathbb{P}\left(\left|\frac{T(X_n)}{\sqrt{n}} - \frac{\mathbb{E}T(X_n)}{\sqrt{n}}\right| \geq t\right) \quad \leq \quad C \exp\left(-C_1 t^4 n\right)$$

Non uniform distribution of nodes

To prove a similar result when the nodes are i.i.d distributed with PDF $f(x)$ we require the following

Theorem (Asymptotic Glueing Lemma, Ganti and Haenggi (2008))

Consider two bounded disjoint sets $A, B \subset \mathbb{R}^2$ and an infinite sequence of nodes $\{x_i\}$. Let $X_n = \{x_1, x_2, \cdots, x_n\}$ be a subset of the sequence. We then have

$$T(X_n \cap A) + T(X_n \cap B) \quad \leq \quad T(X_n \cap (A \cup B)) + o(\sqrt{n}), n \to \infty$$

By a Theorem of Steele and the asymptotic glueing Lemma we have

Theorem

Let $y_i \in \mathbb{R}^2$ be i.i.d random variables , with PDF $f(x)$ (i.e., no singular part w.r.t Lebesgue measure) and bounded support. We then have

$$\lim_{n \to \infty} \frac{T(y_1, y_2, \cdots, y_n)}{\sqrt{n}} = A_2 \int_{\mathbb{R}^2} \sqrt{f(x)} dx$$

- A_2 does not depend on f
- When the nodes are uniformly distributed, observe that we get back the previous result.