On the Iterative Decoding of High Rate LDPC Codes With Applications in Compressed Sensing

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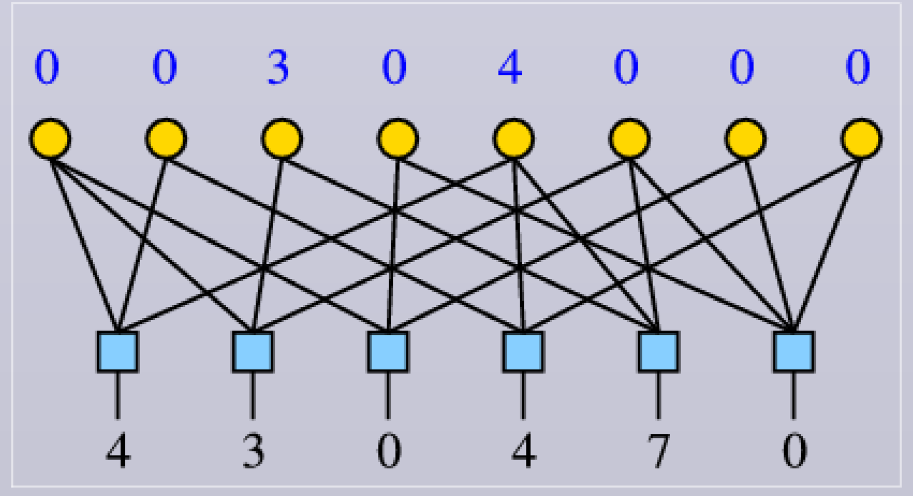
Motivation

Compressed sensing (CS) is very closely related to error correcting codes and can be seen as source coding using linear codes over real numbers. We propose CS sampling/reconstruction algorithms based on low-density parity-check (LDPC) codes with verification decoding. Since most of the interesting applications of CS require very sparse (or compressible) signals, the natural mapping to coding implies high-rate codes. Therefore, we derive the scaling law of the density evolution (DE) [1] analysis and stopping set analysis as the code rate goes to 1. An important implication of this work is that our randomized reconstruction system allows linear-time reconstruction of strictly-sparse signals with a constant oversampling ratio. In contrast, all previous reconstruction methods with moderate reconstruction complexity have an oversampling ratio which grows logarithmically with the signal dimension.

Encoding and Decoding Algorithms

Based on LDPC codes with verification decoding, the encoding algorithm can be described as follows.

1. The observation symbol is the weighted sum (using the edge weights) of the d neighboring signal components as the following figure shows.

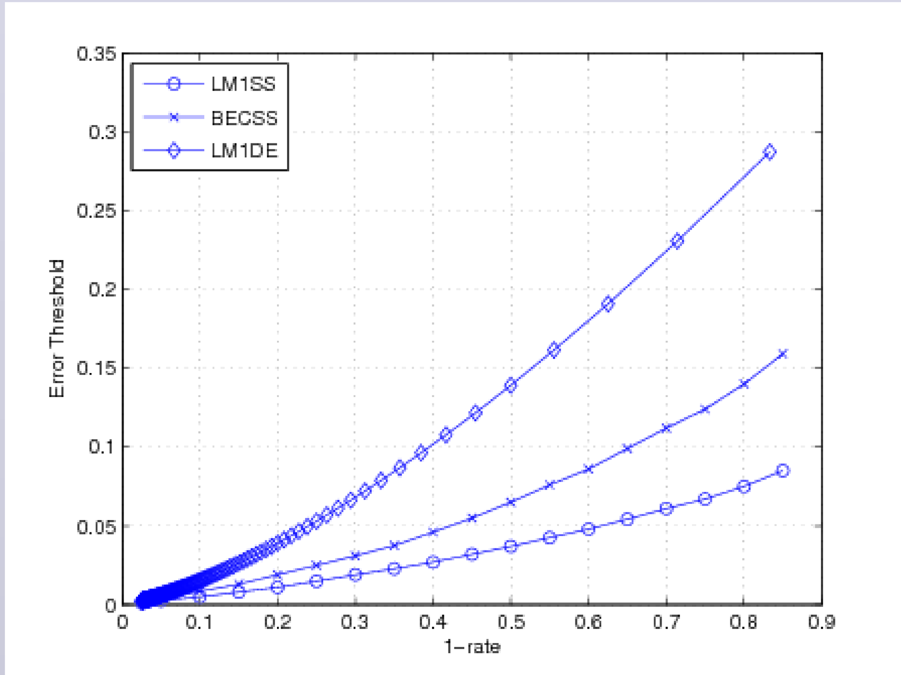


The decoding algorithm can be described as follows.

1. If a measurement is zero, then all the neighboring variable nodes are verified as zero.
2. If a check node is of degree one, verify the variable node with the value of the measurement.
3. [Enhanced verification] If two check nodes overlap in a single variable node and have the same measurement value, then verify that variable node to the value of the measurement.
4. Remove all verified variable nodes and the edges attached to them by subtracting out the verified values from the measurements.
5. Repeat steps 1-4 until decoding succeeds or makes no further progress.

Analysis Tools

Based on the bipartite graph structure, LDPC codes can be decoded efficiently using iterative MP algorithms. The average performance of MP decoding algorithms can be analyzed with DE or extrinsic information transfer (EXIT) charts [3]. In CS systems, this is called *randomized* (or *non-uniform*) reconstruction. Decoding can also be analyzed using combinatorial methods such as stopping set analysis [2] and [4]. In CS systems, this is called *uniform* reconstruction. Another criterion, which is between uniform reconstruction and randomized reconstruction, is what we call *uniform-in-probability* reconstruction. A CS system achieves uniform-in-probability reconstruction if, for any signal in the signal set, *most* randomly chosen measurement matrices achieve successful decoding.
The analysis of this paper gives the following bounds on the decoding thresholds.



Results of DE Scaling Analysis

For the simplicity of our analysis, we only consider (j,k)-regular code ensemble and the LM1 decoding algorithm [5] for the q-SC with error probability δ. The DE recursion for LM1 is (from [5])

$$x_{i+1}=\delta\left(1-\left[1-(1-\delta)\left(1-(1-x_i)^{k-1}\right)^{j-1}-x_i\right]^{k-1}\right)^{j-1}, \quad (1)$$

where x_i is the fraction of unverified messages in the i-th iteration.

Theorem 1 *Consider a sequence of (j,k)-regular LDPC codes with fixed symbol degree $j\geq 2$ and increasing check degree k. Let $\bar{\alpha}_j$ be the largest α such that $(1-e^{-\alpha^{j-1}x^{j-1}})^{j-1}\leq x$ for $x\in(0,1]$. If the sparsity of the signal is $n\delta$ for $\delta=\alpha(k-1)^{-j/(j-1)}$ and $\alpha<\bar{\alpha}_j$, then there exists a K_1 such that by randomly choosing a length-n code from the (j,k) regular LDPC code ensemble, LM1 reconstruction succeeds (w.h.p as $n\to\infty$) for all $k\geq K_1$. Conversely, if $\alpha>\bar{\alpha}_j$ then there exists a K_2 such that LM1 reconstruction fails (w.h.p as $n\to\infty$) for all $k\geq K_2$.*

Theorem 2 *Consider a sequence of (j,k)-regular LDPC codes with variable node degree $j\geq 3$. Let $\bar{\alpha}_j$ be the largest α such that $\left(1-e^{-\alpha jx}\right)^{j-2}\left(1+(j-2)e^{-\alpha jx}\right)\leq x$ for $x\in(0,1]$. If the sparsity of the signal is $n\delta$ with $\delta=\alpha j/k$ and $\alpha<\bar{\alpha}_j$, then there exists a K_1 such that LM2-MB reconstruction succeeds (w.h.p as $n\to\infty$) for all $k\geq K_1$. Conversely, if $\alpha>\bar{\alpha}_j$ then there exists a K_2 such that LM2-MB decoding fails (w.h.p as $n\to\infty$) for all $k\geq K_2$.*

Remark 1 *If a randomly chosen code from the (j,k) regular ensemble is applied to a CS system with LM2-MB reconstruction, then randomized reconstruction succeeds (w.h.p as $n\to\infty$) when the sparsity is $n\delta$ with $\delta<\bar{\alpha}_j j/k$. This requires $m\geq\gamma n\delta$ measurements and an oversampling ratio of $\gamma>\gamma_0=1/\bar{\alpha}_j$.*

Results of Stopping Set Scaling Analysis

In the stopping set analysis for q-SC, we can define $E_{n,j,k}(\alpha,\beta)$ as the *average number of stopping sets* with $|T|=n\alpha$ correctly received variable nodes and $|U|=n\beta$ incorrectly received variable nodes where n is the code length. $E_{n,j,k}$ can be calculated as

$$E_{n,j,k}(\alpha,\beta)=\frac{\binom{n}{n\alpha,n\beta,n(1-\alpha-\beta)}S_{n,j,k}(\alpha n,\beta n)}{\binom{nj}{nj\alpha,nj\beta,nj(1-\alpha-\beta)}} \quad (2)$$

where

$$S_{n,j,k}(a,b)\triangleq \text{coeff}\left(g_k(x,y)^{nj/k},x^{ja}y^{jb}\right).$$

Define the *normalized average stopping set distribution* $\gamma_{j,k}(\alpha,\beta)$ as

$$\gamma_{j,k}(\alpha,\beta)=\lim_{n\to\infty}\frac{1}{n}\ln E_{n,j,k}(\alpha,\beta). \quad (3)$$

We have

Theorem 3 *The normalized average stopping set distribution $\gamma_{j,k}(\alpha,\beta)$ for LM1 can be bounded by*

$$\gamma_{j,k}(\alpha,\beta)\leq\gamma_{j,k}(\alpha,\beta;x,y)\triangleq \frac{j}{k}\ln\frac{\left(1+(1+x+y)^k-ky-(1+x)^k\right)}{x^{k\alpha}y^{k\beta}} +(1-j)h(\alpha,\beta,1-\alpha-\beta). \quad (4)$$

The following theorem shows the scaling law of LM1 for the q-SC.

Theorem 4 *There is a code from (j,k) regular LDPC code ensemble and a constant K such that for the q-SC, all error patterns of size $n\delta$ for $\delta<\bar{\beta}_j(k-1)^{-j/(j-2)}$ can be recovered by LM1 (w.h.p. as $n\to\infty$) for $k\geq K$ where $\bar{\beta}_j$ is the unique positive root on c of the following implicit function*

$$v(d)=\frac{d}{2}\big((c-1)j\ln(1-c)-2c\ln(c) +(1+c)(j-2)(-1+\ln d)\big) \quad (5)$$

where $d=(1-c)^{-j/(j-2)}c^{2/(j-2)}$.

Remark 2 *In a CS system with strictly sparse signals and LM1 reconstruction, we have uniform-in-probability reconstruction (w.h.p. as $n\to\infty$) of all signals with sparsity at most $n\delta$ where $\delta<\bar{\beta}_j(k-1)^{-j/(j-2)}$. This requires $m=\gamma n\delta$ measurements and an oversampling rate of $\gamma>\gamma_0=\bar{\beta}_j^{-(j-2)/j}j\delta^{-2/j}$.*

Remark 3 *If the signal has all non-negative components, then the verification based algorithm will have no FV because the neighbors of a check node will sum to zero only if these neighbors are exactly zero. Therefore, the above analysis implies uniform recovery of non-negative signals that are sufficiently sparse.*

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