Quantization for Compressed Sensing Reconstruction

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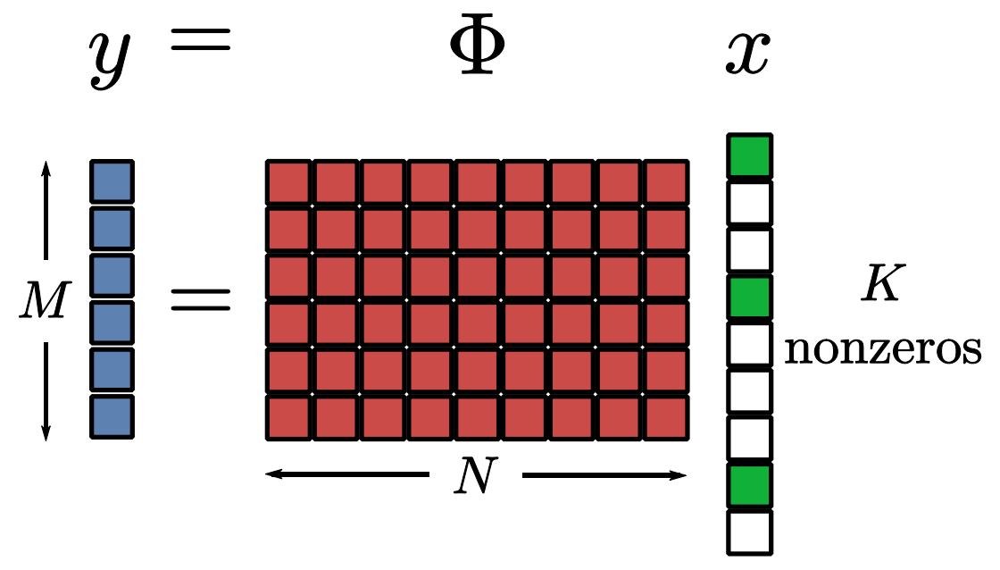
Summary

Since reconstruction algorithms in compressed sensing are nonlinear functions of random measurements, quantization to minimize MSE of reconstruction is not the same as quantization to minimize MSE of measurements. We extend previous work in functional quantization and apply it to a quantized compressed sensing model to find the sensitivity function which defines the optimal quantizer.

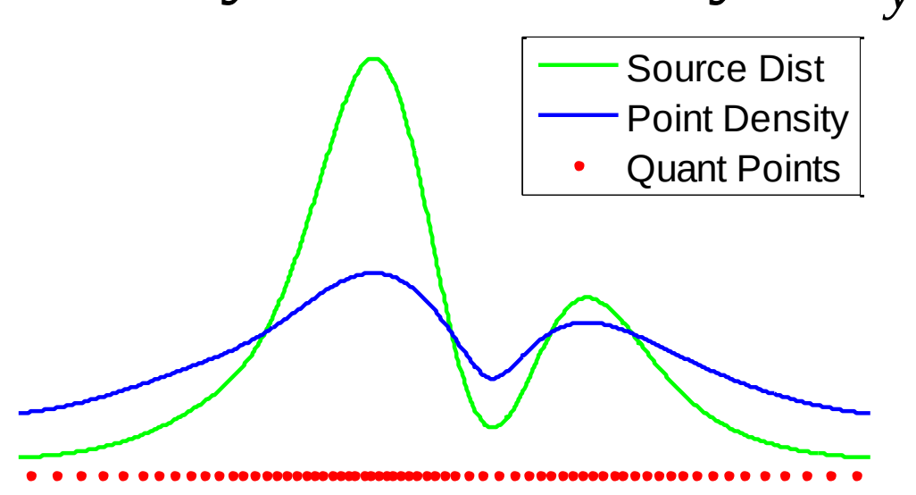
Compressed Sensing

- Emerging non-adaptive compression technique
- *K*-sparse input vector *x*
- Random sampling matrix **Φ**
- Can use *nonlinear* reconstruction method like lasso or OMP to find best sparse solution

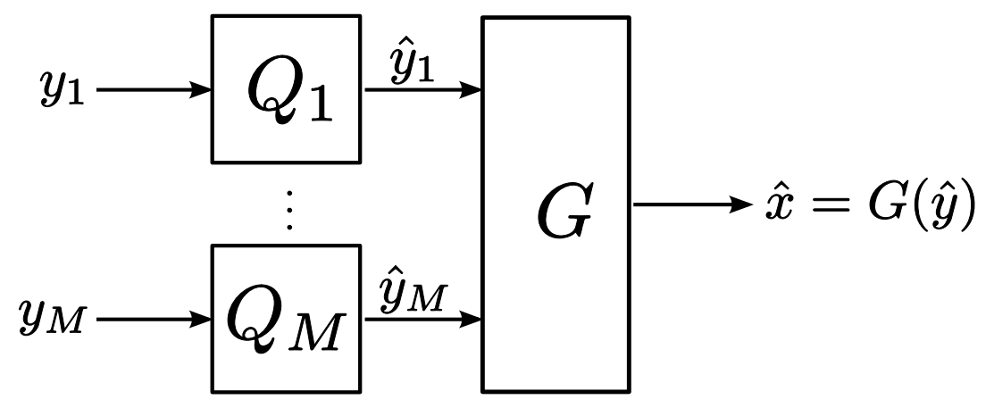


Functional Quantization

- The point density is high-res approximation of quantizer s.t.
  $\Delta\lambda(t)$: approx. fraction of points in $[t–½\Delta, t–½\Delta]$
- The optimal point density for a source y is $\lambda_y(t) = (f_{\mathbf{y}}(t))^{1/3}$



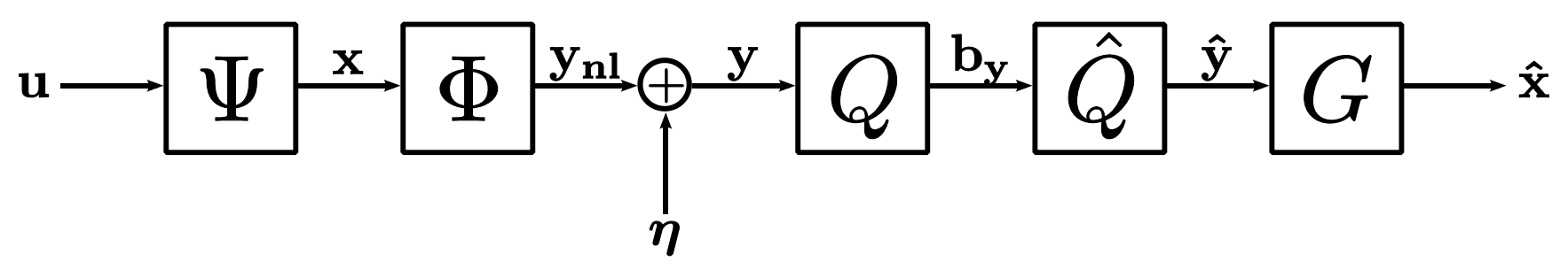
- In functional quantization, optimize to minimize error of some function of a source rather than the source itself



- If $\lambda_y(t)$ is best quantizer for source vector **y**, then optimal functional quantizer for $\mathbf{y_i}$ is $\lambda_i(t) = \lambda_{\mathbf{y_i}}(t) \cdot \gamma_i^2(t)$
- $\gamma_i(t)$ is the sensitivity profile of the function *G* and is defined as

$$\gamma_i^2(t) = E\left[\left|\frac{\partial g(\mathbf{y})}{\partial \mathbf{y_i}}\right|^2 \mid \mathbf{y_i} = t\right]$$

Problem Model



- Sparsity pattern **J** chosen uniformly
- Nonzero elements $\mathbf{x_j}$ ~ *U(-1,1)*
- Random matrix **Φ** chosen s.t. columns have unit norm
- Fixed-rate quantizer *Q* has total rate *R*
- Reconstruction method *G* is lasso, a quadratic program

$$\hat{x} = \arg\min_x \left(\|y - \Phi x\|_2^2 + \mu\|\Psi^{-1}x\|_1\right)$$

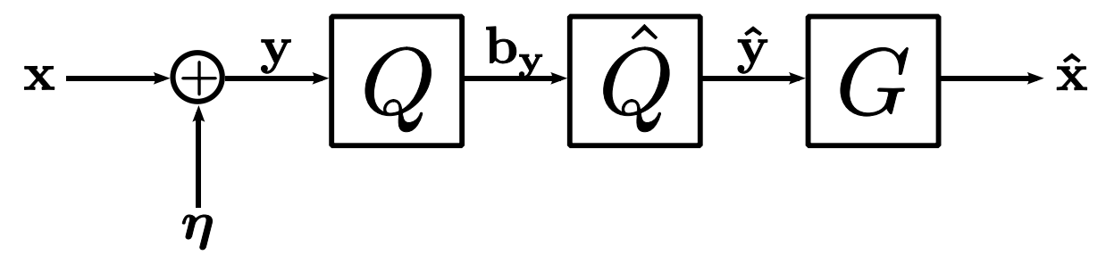
To find partial derivatives for lasso reconstruction, we use the *homotopy continuation* method. Using this method, we scan μ from a large value to 0 to find the sparsity and the resulting reconstruction at each step. The equation for each μ^* and its corresponding sparsity satisfies:

$$2\Phi_{J_{\mu^*}}^T \Phi_{J_{\mu^*}} \hat{x} + \mu^* \operatorname{sgn}(\hat{x}) = 2\Phi_{J_{\mu^*}}^T y$$

Resulting differentials are

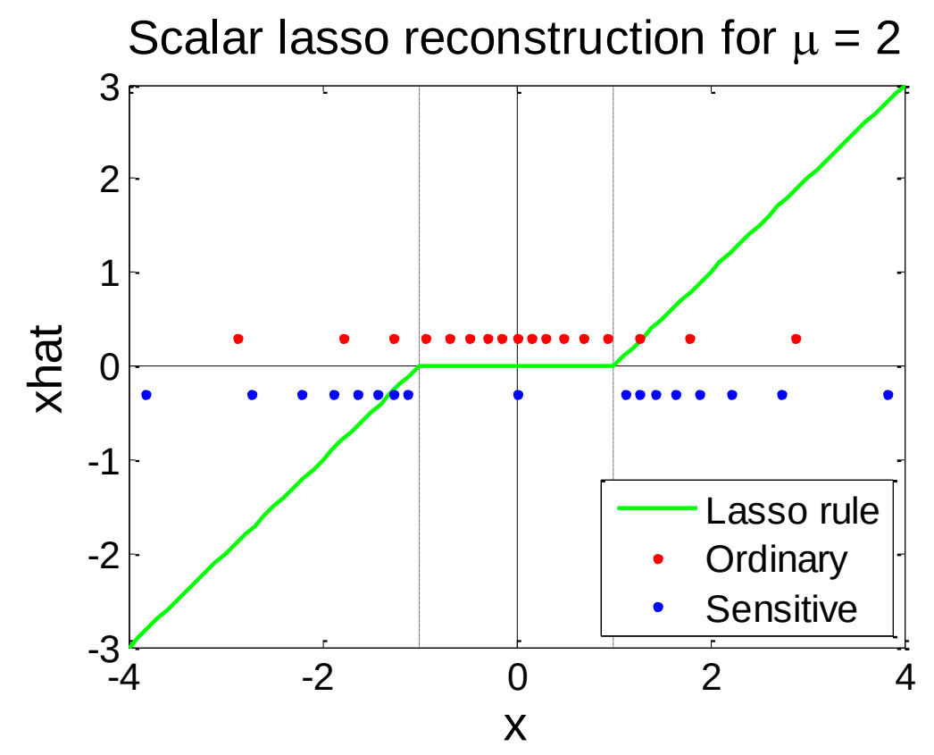
$$G_i^{(j)}(y, \Phi) = \frac{\partial G^{(j)}(y,\Phi)}{\partial y_i} = \left[\left(\Phi_{J_{\mu^*}}^T \Phi_{J_{\mu^*}}\right)^{-1} \Phi_{J_{\mu^*}}^T\right]_{ji}$$

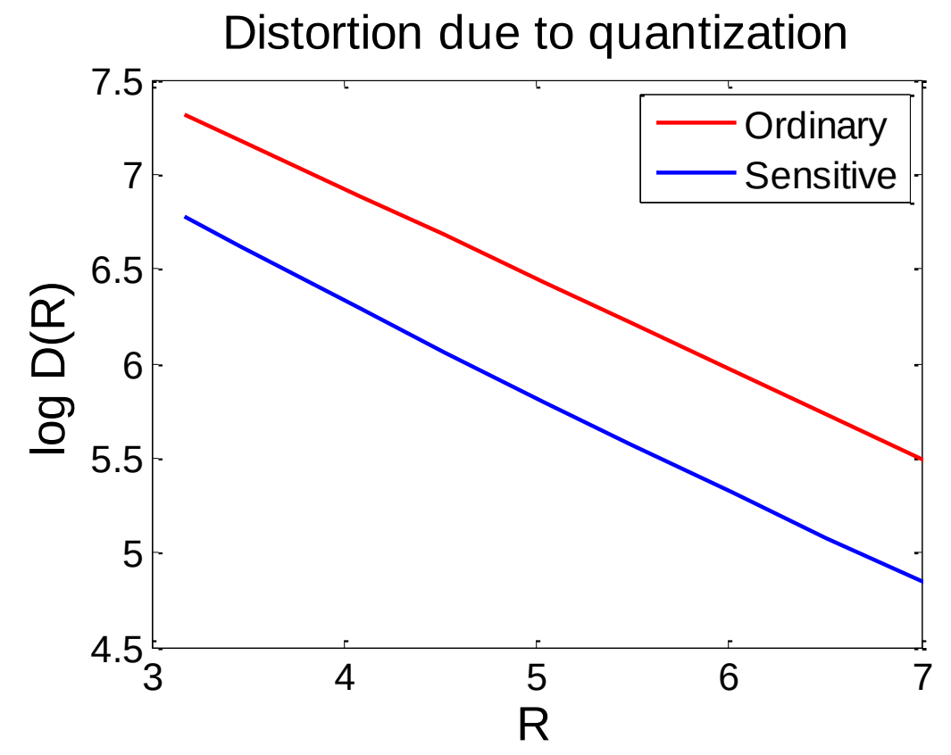
Motivating Example



- Scalar **x** ~ *Lap(1)*
- Noise **η** ~ *N(0,0.1)*
- Reconstruction method is scalar lasso:

$$\hat{x} = \arg\min_x \left(\|y - x\|_2^2 + \mu\|x\|_1\right)$$





- We note a *1dB* improvement from the sensitive quantizer

Quantizer Design

We prove

$$\gamma_i^{(j)}(t) = \left(E_{\boldsymbol{\Phi}}\left[\frac{f_{\mathbf{y_i}|\boldsymbol{\Phi}}(t|\boldsymbol{\Phi})}{f_{\mathbf{y_i}}(t)} E_{\mathbf{y}_{\setminus\mathbf{i}}}\left[\left|G_i^{(j)}(\mathbf{y}, \Phi)\right|^2 \mid \mathbf{y_i} = t, \boldsymbol{\Phi}\right]\right]\right)^{\frac{1}{2}}$$

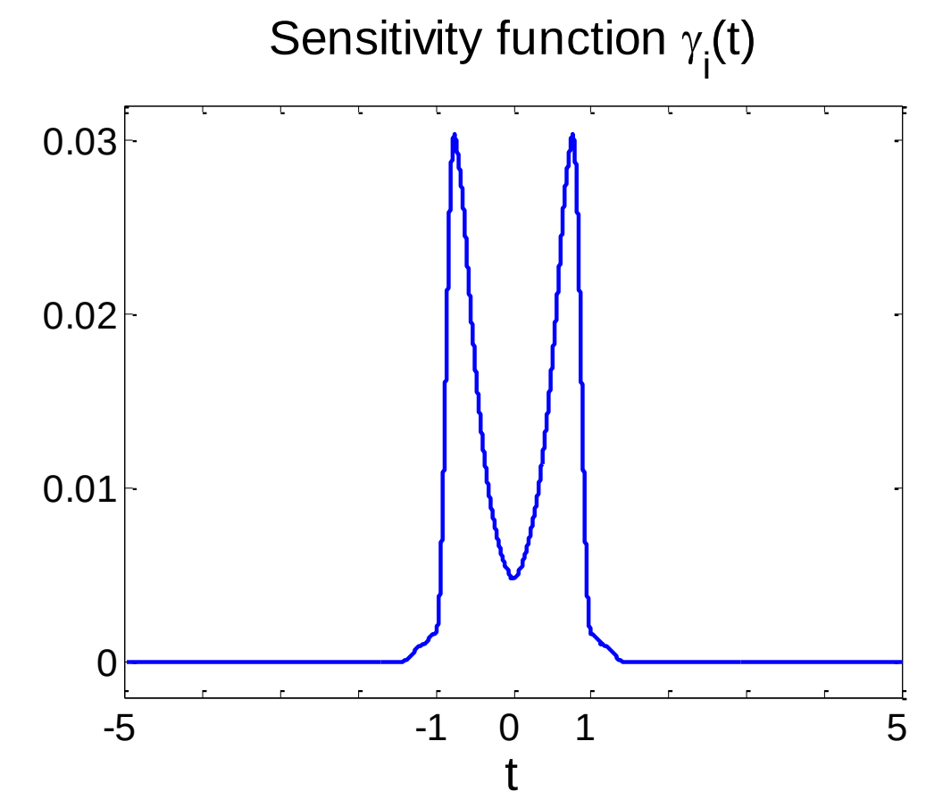
We approximate sensitivity through Monte Carlo trials:

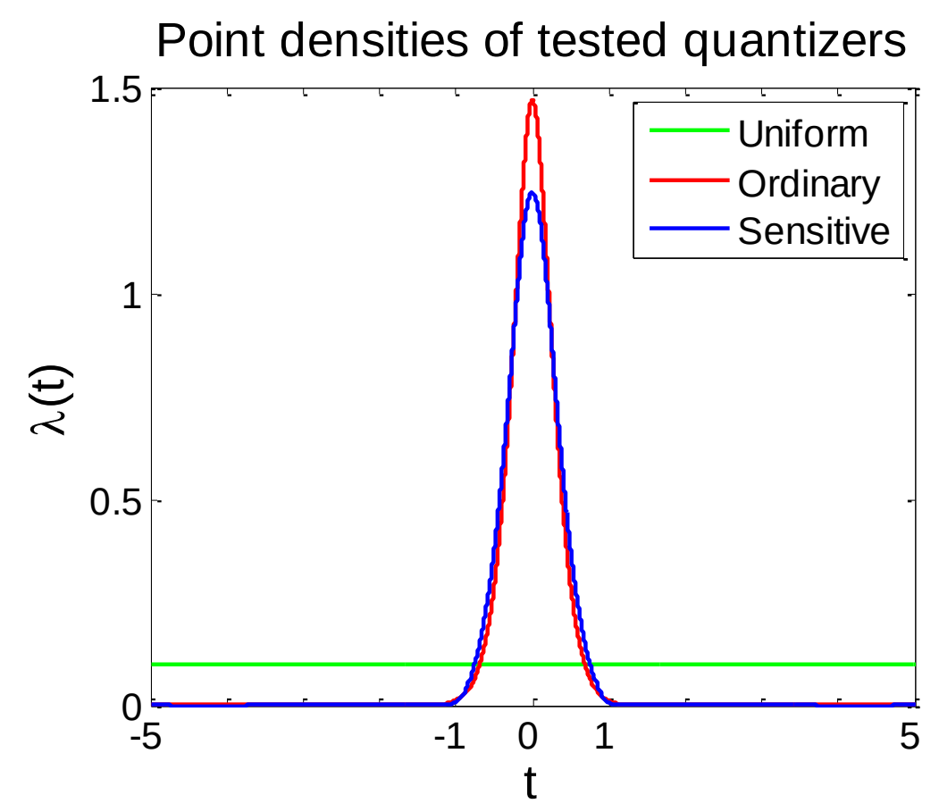
$$\gamma_{cs}^{(L)}(t) = \frac{1}{L}\sum_{\ell=1}^{L}\left(\frac{f_{\mathbf{y_i}|\boldsymbol{\Phi}}(t|\Phi_\ell)}{f_{\mathbf{y_i}}(t)}\left[G_i^{(j)}(y, \Phi_\ell)\right]^2\right)^{\frac{1}{2}}$$

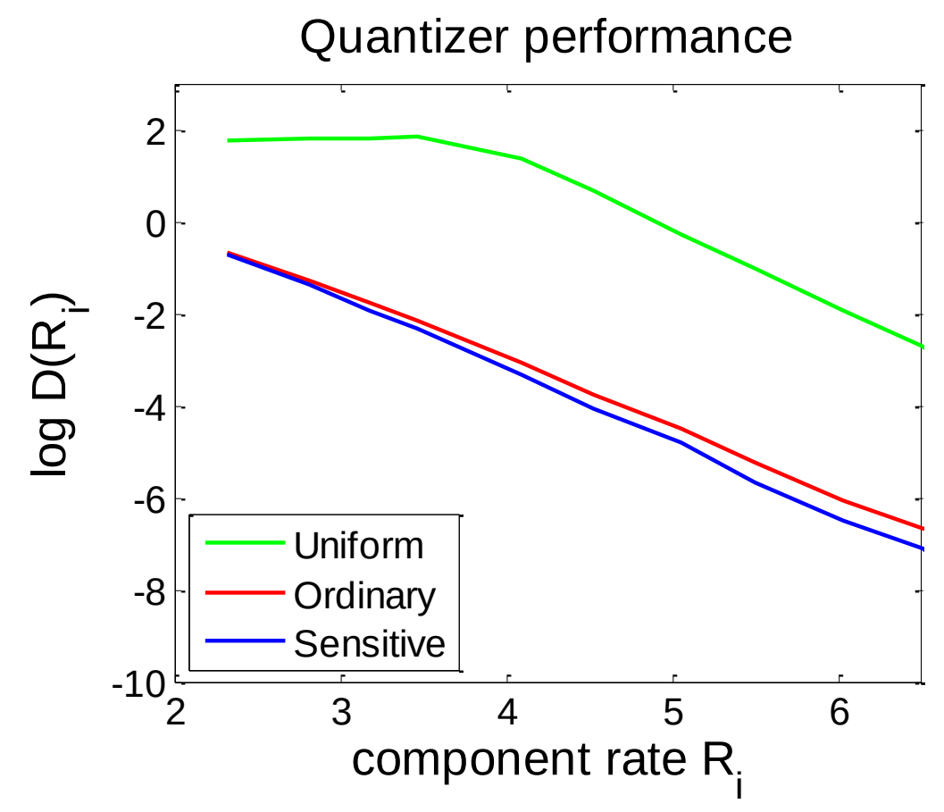
The quantizer is found using HR approximation and approaches the true best quantizer:

$$\lambda_{cs}^{(L)}(t) = C\left(\gamma_{cs}^{(L)}(t) f_{\mathbf{y_i}}(t)\right)^{1/3} \xrightarrow{p} \lambda_{cs}(t)$$

Results







- We find experimental sensitivity is not flat and peaks away from zero due to structure of lasso
- The optimal fixed-rate quantizer is better than a uniform one by *6dB* and Lloyd-Max by about *1dB*
- Optimal variable-rate quantizer can be found similarly

References

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- V. Misra, V. K. Goyal, and L. R. Varshney, "Distributed functional scalar quantization: High-resolution analysis and extensions," arXiv:0811.3617v1 [cs.IT]., Nov. 2008.
- D. M. Malioutov, M. Cetin, and A. S. Willsky, "Homotopy continuation for sparse signal representation," in *Proc. IEEE Acoustics, Speech and Sig. Proc. Conf.*, Mar. 2006.