A New Upper Bound on the Maximal Error Exponent for Multiple-Access Channels

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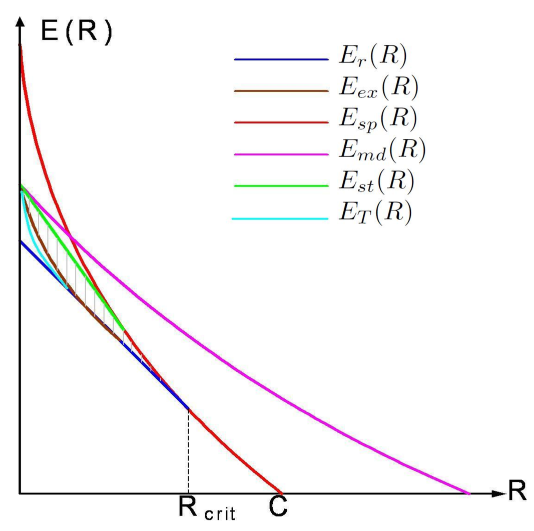
1. Point to Point: Bounds on the Reliability Function

$E(R)$: Rate of exponential decay of codeword error probability



$$E_L(R) = \max_{P\in\mathcal{P}_L} \min_{V\in\mathcal{V}_L} f_L(R,P,V)$$
$$E_U(R) = \max_{P\in\mathcal{P}_U} \min_{V\in\mathcal{V}_U} f_U(R,P,V)$$

- Why maximization in Lower bounds?
  - Freedom in choosing any constant composition code
- Why maximization in Upper bounds?
  - For all codes, there must be a dominant type; However, it is unknown

$$\mathcal{P}_L = \mathcal{P}_U = \{P : H(P) \geq R\}$$

2. MAC: Bounds on the Reliability Function

- Few results compared to point-to-point
- **No known rate region over which the bounds are tight**

$$E_L = \max_{P\in\mathcal{P}_L} \min_{V\in\mathcal{V}_L} f_L(R_X,R_Y,P,V) \qquad E_U = \max_{P\in\mathcal{P}_U} \min_{V\in\mathcal{V}_U} f_U(R_X,R_Y,P,V)$$

- Why maximization in Lower bounds?
  - Not that much freedom in comparison with point to point case

$$\mathcal{P}_L = \{P_{UXY} : X-U-Y \quad R_X \leq H(X|U) \;\;\&\;\; R_Y \leq H(Y|U)\}$$

- Why maximization in Upper bounds?
  - For all codes, there must be a dominant type; However, it is unknown

$$\mathcal{P}_U = \{All \;\; P_{XY}\}$$

3. Our Contribution

- Reducing the set $\mathcal{P}_U$ to $\mathcal{P}_L$

$$\mathcal{P}_U = \mathcal{P}_L = \{P_{UXY} : X-U-Y \quad R_X \leq H(X|U) \;\;\&\;\; R_Y \leq H(Y|U)\}$$

- A new upper bound on the reliability function
  - Minimum distance bound
- Comparison with the expurgated bound for MAC
  - Similar structure at zero rate

4. Minimum distance bound: Steps of derivation

- Choose some distance function:**Bhattacharyya distance**

$$d_B\big((x,y),(\tilde{x},\tilde{y})\big) \triangleq -\log\Big(\sum_{z\in\mathcal{Z}} \sqrt{W(z|x,y)W(z|\tilde{x},\tilde{y})}\Big)$$

  - *Indivisible Channel*: $d_B\big((x,y),(\tilde{x},\tilde{y})\big) \neq \infty$
  - *Nonnegative-definite Channel*: For all $s>0$, the matrix $A_{(i,j),(k,l)} = 2^{sd_B((i,j),(k,l))}$ is nonnegative-definite.
  - Normalized Bhattacharyya distance (DM-MAC)

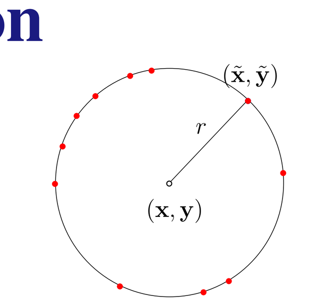
$$d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big) = \sum_{\substack{x,\tilde{x}\in\mathcal{X}\\ y,\tilde{y}\in\mathcal{Y}}} P_{\mathbf{x}\mathbf{y}\tilde{\mathbf{x}}\tilde{\mathbf{y}}}(x,y,\tilde{x},\tilde{y})\, d_B\big((x,y),(\tilde{x},\tilde{y})\big)$$

- Show that there exist at least two pairs of codewords which are very close to each other.
  - Find an upper bound on the minimum distance of codes with fixed rate pairs and blocklength (n,R_X,R_Y)
- Make a connection between the minimum distance of the code and the maximal probability of decoding error
  - The closer the sequences are, the more confusion in decoding
- Use the upper bound on Bhattacharyya distance to infer the lower bound on probability of decoding error

5. Spherical Collection

- $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$
- $(\mathbf{x},\mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n$, $V : \mathcal{X}\times\mathcal{Y} \to \mathcal{X}\times\mathcal{Y}$

$$S_{XY}(\mathbf{x},\mathbf{y}) = \{(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in \mathcal{C} : (\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \in T_V(\mathbf{x},\mathbf{y})\}$$
$$r = d_B\big((\mathbf{x},\mathbf{y}),(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\big)$$



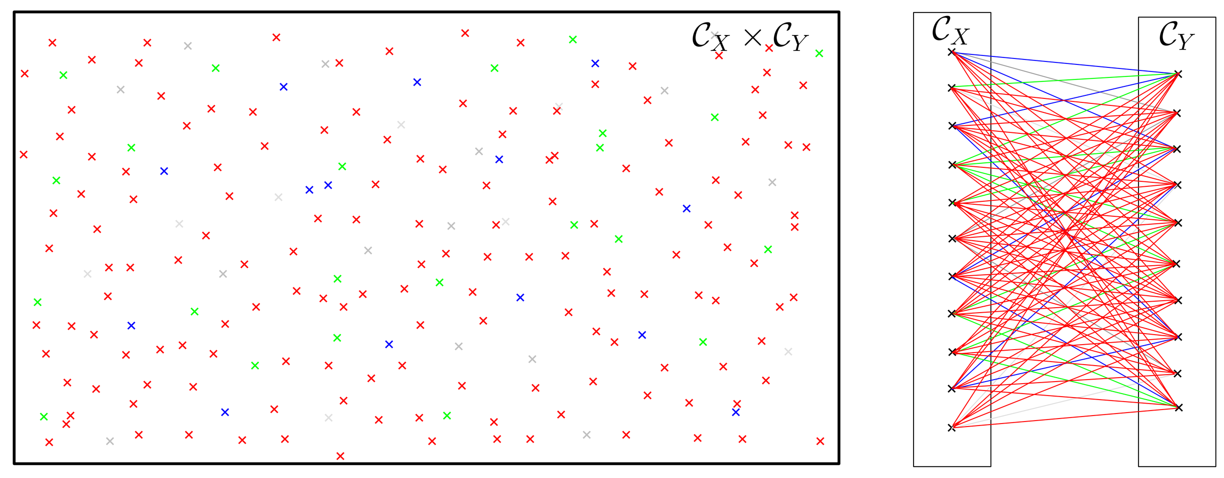
Sperical Collection about $(\mathbf{x},\mathbf{y})$ defined by V

Goal: For any V satisfying certain condition, there exist a pair of sequences, not necessarily codewords, such that the spherical collection about it, contains exponentially many codeword pairs.

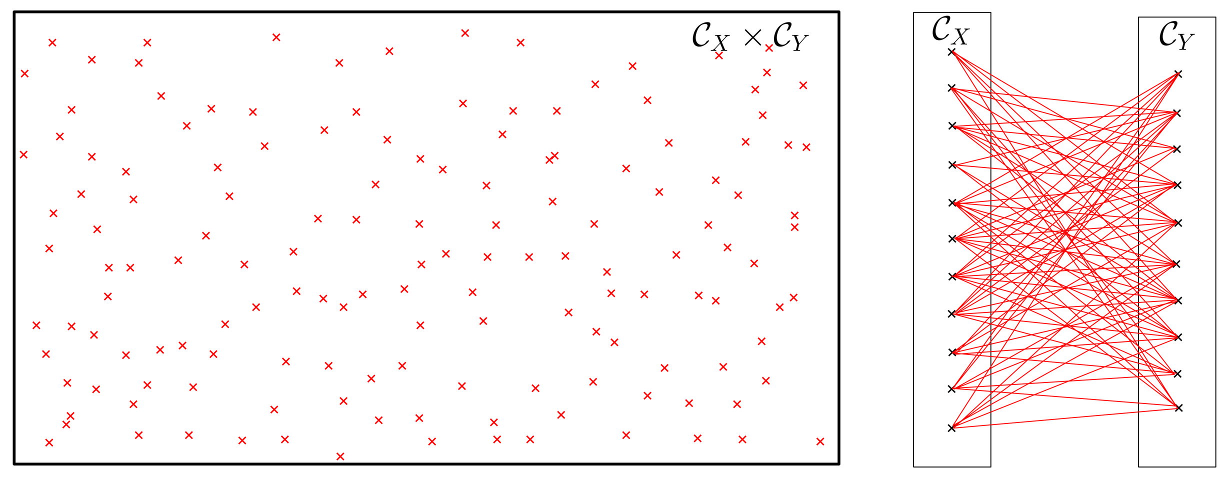
- All of these pairs cannot be far from each other.

- A. Nazari, S. S. Pradhan, A. Anastasopoulos, New Bounds on the Maximal Error Exponent for Multiple-Access Channels, ISIT 2009, Seoul, South Korea. (IEEE Information Theory Society Best Student Paper Award)

6. Dominant type of the codebook



- Number of codeword pairs is an exponential function of n
- Number of joint type is a polynomial function of n
- An ***almost fully connected*** subcode

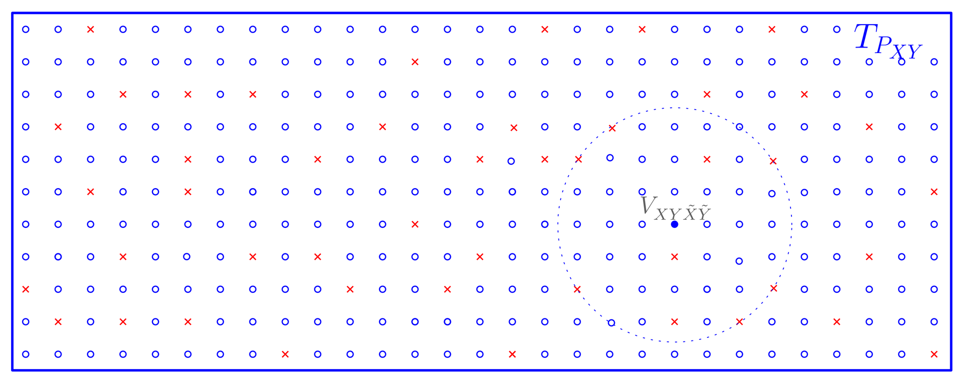


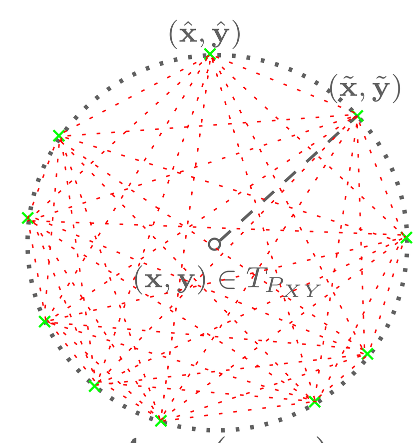
The dominant type theorem: For a code with parameter (R_X,R_Y), a joint type P_{XY} can be the dominant type, if for some $P_{U|XY}$ satisfying $X-U-Y$

$$R_X \leq H(X|U) \qquad \& \qquad R_Y \leq H(Y|U)$$

7. Dense Spherical Collection

- P_{XY}: A dominant type of $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ $\quad \& \quad V_{XY\tilde{X}\tilde{Y}}$: $V_{XY} = V_{\tilde{X}\tilde{Y}} = P_{XY}$
- By a counting argument $\Rightarrow \exists\, (\mathbf{x},\mathbf{y}) \in T_{P_{XY}}\; \mathcal{A}_{XY} \geq \exp\{n[R_X+R_Y-I(\tilde{X}\tilde{Y}\wedge XY)]\}$





$$\mathcal{A}_{XY}(\mathbf{x},\mathbf{y})$$
$$(\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}},\hat{\mathbf{x}},\hat{\mathbf{y}}) \in T_{V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}}}$$
$$V_{XY} = V_{\tilde{X}\tilde{Y}} = V_{\hat{X}\hat{Y}} = P_{XY}$$
$$V_{\tilde{X}\tilde{Y}|XY} = V_{\hat{X}\hat{Y}|XY}$$
$$\tilde{X}\tilde{Y} - XY - \hat{X}\hat{Y}$$
$$I_V(XY \wedge \tilde{X}\tilde{Y}) \leq R_X + R_Y$$

- Assume $I(\tilde{X}\tilde{Y} \wedge XY) < R_X + R_Y$
- Study the distance structure of $\mathcal{A}_{XY}(\mathbf{x},\mathbf{y})$
  - Find the average distance of this subset
- Exponentially many codeword pairs in $\mathcal{A}_{XY}(\mathbf{x},\mathbf{y}) \Rightarrow$ all of them cannot be far from each other. $\exists(\tilde{\mathbf{x}},\tilde{\mathbf{y}}),(\hat{\mathbf{x}},\hat{\mathbf{y}}) \in \mathcal{C}$

$$d_B\big((\tilde{\mathbf{x}},\tilde{\mathbf{y}}),(\hat{\mathbf{x}},\hat{\mathbf{y}})\big) \leq Ed_B\big((\tilde{X},\tilde{Y}),(\hat{X},\hat{Y})\big)$$

8. MAC: Minimum Distance Bound

Theorem: For any nonnegative-definite channel, W,

$$E^*_m(R_X,R_Y) \leq d^*_B(R_X,R_Y) \leq E_U(R_X,R_Y,W) \triangleq \max_{P_{UXY}} \min_{\beta=X,Y,XY} E^\beta_U(R_X,R_Y,W,P_{XY})$$

- All P_{UXY}: $X-U-Y$, $R_X \leq H(X|U)$, and $R_Y \leq H(Y|U)$

$$\mathbf{E^X_U} \triangleq \min_{\substack{V_{XY\tilde{X}\hat{X}}:\\ V_{\tilde{X}Y}=V_{\hat{X}Y}=V_{XY}=P_{XY}\\ \hat{X}-XY-\tilde{X}\\ V_{\tilde{X}|XY}=V_{\hat{X}|XY}\\ I(X\wedge\tilde{X}|Y)\leq R_X}} Ed_B\big((\hat{X},Y),(\tilde{X},Y)\big) \qquad \mathbf{E^Y_U} \triangleq \min_{\substack{V_{XY\tilde{Y}\hat{Y}}:\\ V_{X\tilde{Y}}=V_{X\hat{Y}}=V_{XY}=P_{XY}\\ \hat{Y}-XY-\tilde{Y}\\ V_{\tilde{Y}|XY}=V_{\hat{Y}|XY}\\ I(Y\wedge\tilde{Y}|X)\leq R_Y}} Ed_B\big((X,\hat{Y}),(X,\tilde{Y})\big)$$

$$\mathbf{E^{XY}_U} \triangleq \min_{\substack{V_{XY\tilde{X}\tilde{Y}\hat{X}\hat{Y}}:\\ V_{\tilde{X}\tilde{Y}}=V_{\hat{X}\hat{Y}}=V_{XY}=P_{XY}\\ \hat{X}\hat{Y}-XY-\tilde{X}\tilde{Y}\\ V_{\tilde{X}\tilde{Y}|XY}=V_{\hat{X}\hat{Y}|XY}\\ I(XY\wedge\tilde{X}\tilde{Y})\leq R_X+R_Y}} Ed_B\big((\hat{X},\hat{Y}),(\tilde{X},\tilde{Y})\big)$$

9. Conclusions

- Bringing the auxiliary random variable U into the expression for all upper bounds (e.g. Sphere Packing, Minimum distance bounds)
  - Similar to the lower bounds
- A new upper bound on the reliability function of MAC
  - Tighter than sphere packing at low rate regime
- Comparison with expurgated bound
  - Similar structure specially at zero rate

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For references, questions and comments please contact anazari@umich.edu