On the Capacity of Single-Relay Channel with ISI

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Problem Statement(Linear Gaussian Relay Channel)

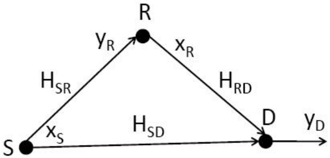


Figure: Gaussian relay channel

- Channel outputs with memory m

  $y_{Rk} = \sum_{i=0}^{m} h_{SRi} x_{S(k-i)} + v_{Rk}, y_{Dk} = \sum_{i=0}^{m} (h_{SDi} x_{S(k-i)} + h_{RDi} x_{R(k-i)}) + v_{Dk}$

- Additive non-white Gaussian noise with memory $\leq m$
- Power constraints at source and relay

$$\frac{1}{n} \sum_{k=1}^{n} E[x_{qk}^2] \leq P_q, \quad q \in \{S, R\} \ \forall \ n$$

- Physically degraded relay channel

$$x_{Sk-m}^{k} \rightarrow \{x_{Rk-m}^{k}, y_R(k)\} \rightarrow y_D(k)$$

Motivation

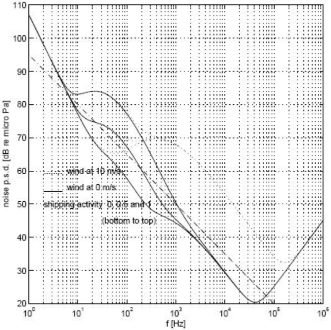


Figure: Noise PSD in UW Acoustic Communication

- Practical communication systems approximately time and bandwidth limited $\longrightarrow$ finite memory
- *e.g.* an underwater relay network, each link of different bandwidths due to distance dependent attenuation
- Non-white Gaussian noise in underwater communication

Capacity of Gaussian Relay Channel with ISI

- [Marina, Kavčić, Gaarder 08]: Capacity (C) of the degraded LGRC with memory m,

$$\begin{aligned} C_{LGRC} &= \lim_{n\to\infty} C_{LGRCn}(P_S, P_R) \\ &= \lim_{n\to\infty} \sup_{p(x_S^n, x_R^n)} \frac{1}{n} \min\{I(x_S^n; y_R^n | x_R^n), I(x_S^n, x_R^n; y_D^n)\} \end{aligned}$$

  subject to power constraints

- Not single-letter, difficult to compute the limit
- Standard circular Gaussian relay channel (CGRC) model, equivalent to our model in terms of capacity
- The input-output relation:

$$\begin{aligned} \bar{y}_{Rk} &= \sum_{i=0}^{m} h_{SRi} x_{S(k-i)_n} + \bar{v}_{Rk} \\ \bar{y}_{Dk} &= \sum_{i=0}^{m} (h_{SDi} x_{S(k-i)_n} + h_{RDi} x_{R(k-i)_n}) + \bar{v}_{Dk} \end{aligned}$$

- Circulant channel impulse response matrix
- Noise covariance matrices also circulant([Goldsmith, Effros 01]):

LGRC vs CGRC

- n-CGRC is a degraded n-block memoryless channel:
- Capacity follows from [Cover, El Gamal 79] by replacing (X, X_1, Y_1, Y) by $(x_S^n, x_R^n, \bar{y}_R^n, \bar{y}_D^n)$

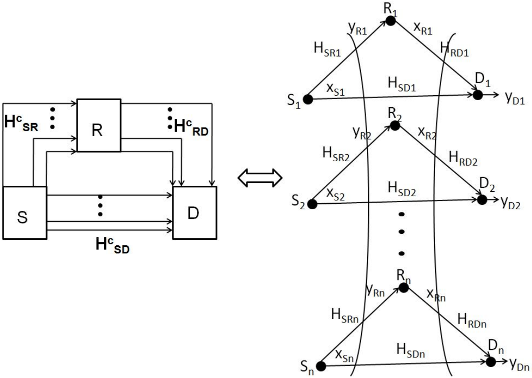
$$
\begin{aligned}
C_{CGRCn}(P_S, P_R) &= \sup_{p(x_S^n, x_R^n)} \frac{1}{n} \min\{I(x_S^n; \bar{y}_R^n | y_R^n), I(x_S^n, x_R^n; \bar{y}_D^n)\} \\
&= \sup_{p(x_S^n, x_R^n)} \frac{1}{n} \min\{C_{1n}, C_{2n}\}
\end{aligned}
$$

Theorem

The capacities of the n-CGRC and LGRC are related by,

$$
\begin{aligned}
\left(1 - \tfrac{m}{n}\right) C_{LGRCn-m}\left(\tfrac{nP_S}{n-m}, \tfrac{nP_R}{n-m}\right) &\leq C_{CGRCn}(P_S, P_R) \\
&\leq \left(1 + \tfrac{m}{n}\right) C_{LGRCn+m}\left(\tfrac{nP_S}{n+m}, \tfrac{nP_R}{n+m}\right)
\end{aligned}
$$

Capacity of n-block CGRC (Decomposition of the Network)



- The n-block degraded CGRC with ISI $=$ set of n-parallel, memoryless and independent scalar relay channels in DFT domain
- Input codewords, which are white across the bandwidth, is optimal
- Permuting the channels via channel matching (e.g. [*Zhang*, *Mitra*08] for the multihop channel) is sub-optimal

Capacity of LGRC

- Decoupling simplifies the capacity expression

Theorem

The capacity of degraded LGRC with memory m is given by,

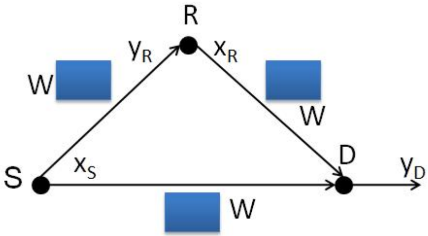
$$\begin{aligned} C_{LGRC}(P_S, P_R) &= \lim_{n\to\infty} C_{CGRCn}(P_S, P_R) \\ &= \max_{\substack{0\le\alpha(\omega)\le 1 \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} P_S(\omega)d\omega \le P_S \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} P_R(\omega)d\omega \le P_R}} \min\{C_1, C_2\} \end{aligned}$$

where,

$$C_1 = \frac{1}{4\pi}\int_{-\pi}^{\pi} C\left(\frac{\alpha(\omega)|h_{SR}(\omega)|^2 P_S(\omega)}{N_R(\omega)}\right) d\omega, \; C_2 = \frac{1}{4\pi}\int_{-\pi}^{\pi} C\left(\frac{P(\omega)}{N_D(\omega)}\right) d\omega$$

- Supremum is performed over power allocation across all the parallel sub-channels

Illustrative Examples



- All the channels same ideal low-pass filters of bandwidth W
- Noises are AWGN:

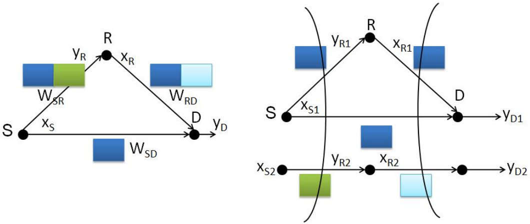
$$N_R(\omega) = N_1, N_D(\omega) = N_1 + N_2 = N,\ 0 \leq \omega \leq \frac{W}{2\pi}$$

- Channel capacity:

$$C = \max_{0 \leq \alpha \leq 1} \min W \left\{ C\left(\frac{\alpha P_S}{N_1 W}\right), C\left(\frac{P_S + P_R + 2\sqrt{\bar{\alpha} P_S P_R}}{NW}\right) \right\}$$

- Uniform input PSD's achieve capacity
- Generalization of the capacity of discrete Gaussian relay channel [Cover, El Gamal 79] to the bandwidth limited case

Links with different bandwidths



- Common in underwater communication where channel bandwidth depends on internode seperation
- All channels are ideal lowpass filters
- Noises have same ideal band-limited PSD as in the earlier example
- The channel is degraded and

$$
\begin{aligned}
C &= \max\min\Big\{ W_{SD} C\left(\frac{\alpha P_{S1}}{N_1 W_{SD}}\right) + (W_{SR} - W_{SD}) C\left(\frac{\alpha P_{S2}}{N_1 (W_{SR} - W_{SD})}\right), \\
&\qquad W_{SD} C\left(\frac{P_{S1} + P_{R1} + 2\sqrt{\bar{\alpha} P_{S1} P_{R1}}}{N W_{SD}}\right) + (W_{RD} - W_{SD}) C\left(\frac{\alpha P_{R2}}{N (W_{RD} - W_{SD})}\right)\Big\}
\end{aligned}
$$

where, $P_{S1} + P_{S2} \leq P_S, P_{R1} + P_{R2} \leq P_R$

Conclusions

- The single-letter capacity of the degraded relay channel with ISI
- The capacity is same as that of a set of parallel memoryless relay channels in DFT domain, illustrated by few examples
- Optimality of OFDM signaling
- Optimal achievability schemes for memoryless relay extends to relay with finite memory
- Future research
  - Investigate optimality of decomposition for non-degraded relay channels
  - Derive analytical solutions for optimal input PSDs that achieve capacity of degarded relay [Submitted to Wuwnet 09]