Compressive Sensing in the Limit

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Goal

Study the asymptotic behaviour of iterative recovery algorithms in the context of compressive sensing over random regular bipartite graphs.

Motivation

Why Asymptotic Results:
1- Gives insight about the finite length performance of algorithms.
2- Analysis is much simpler than the finite length analysis.

Why Iterative Algorithms:
Have computational complexity linear with n, length of the signal.

Why Sparse Graphs:
Encoding and decoding are computationally less complex.

Notations and Definitions

Compressive Sensing System
1- Encoder:
Represents a signal of length n having k nonzero elements with m measurements, where $k < m << n$.

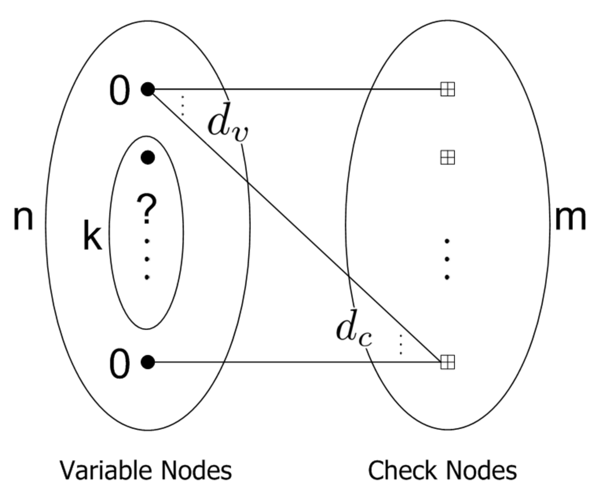
2- Decoder:
Tries to recover back the original signal based on the measurements.

Compression Ratio

$$r_c = \frac{n}{m} = \frac{n}{n\frac{d_v}{d_c}} = \frac{d_c}{d_v}$$

Oversampling Ratio

$$r_o = \frac{m}{k} \rightarrow \frac{d_v}{\alpha d_c}$$



α : Probability of a nonzero variable node

Asymptotic
1- Number of iterations performed is unlimited
2- The ratio k/n is fixed and n tends to infinity

Assumptions

1- The mapping between signal elements and measurements is linear.
Consequence:
The mapping can be represented pictorially using bipartite graphs.

2- Graphs are randomly generated with no parallel edges.
Consequence:
The analysis is made simple.

3- The value of nonzero variable nodes is chosen from a continuous distribution.
Consequence:
If two check nodes have the same value, they are adjacent to the same nonzero variable nodes.

Iterative Recovery Algorithms

The number and location of nonzero variable nodes are not provided to recovery algorithms, except to the Genie.

Pre-Phase
From the regular graph, remove all zero-valued check nodes, their neighbouring variable nodes and all the edges in between.

XH [5,6]
Change the value of a variable node if more than half of its neighbouring check nodes have the same value

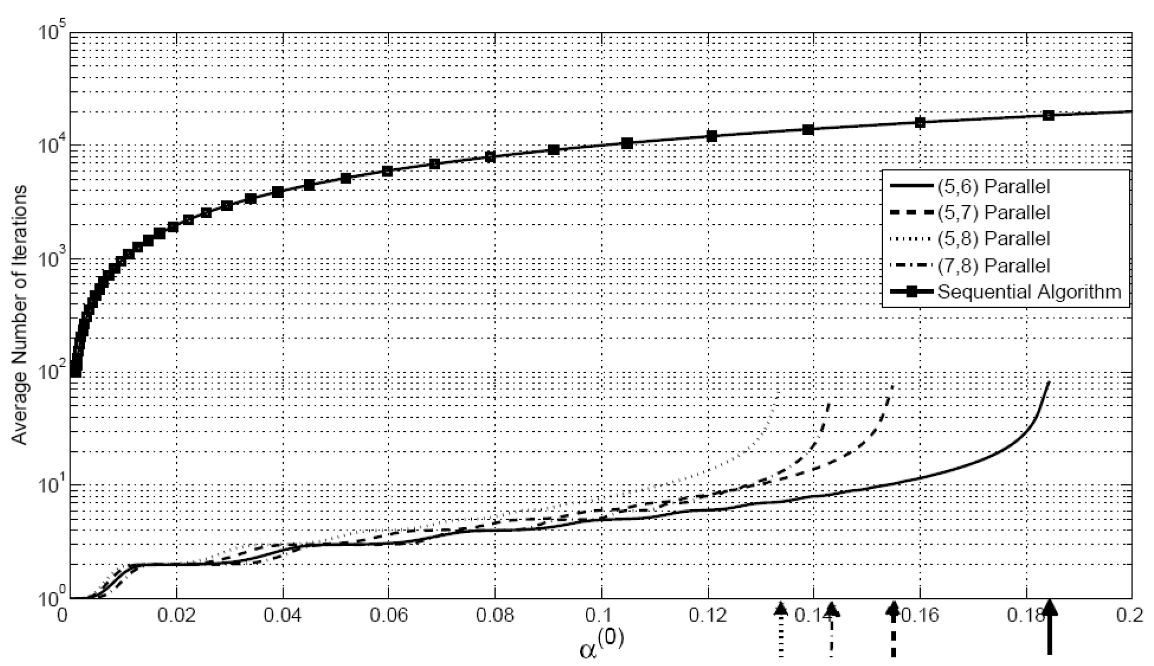
SBB [7]
If two check nodes have the same value:
1- set to zero all variable nodes adjacent to either check nodes
2- resolve the variable node which is the only one adjacent to both check nodes

LM [8]
Change the value of a variable node if it has at least one check node of degree one

Genie
Change the value of a variable node if it has at least one check node of degree one

Summary of Result

1- There is a limiting **threshold** for k/n before which the recovery is successful and beyond which it fails almost surely.
2- The threshold can be calculated **mathematically** by tracking the probability of having a nonzero variable node with iteration.
3- The asymptotic analysis of iterative recovery algorithms follows a **general framework**.
4- Using the analysis, the evolution of unresolved variable nodes can be tracked with iteration number as well as initial sparsity ratio.
5- Finite length results are in a good agreement with the asymptotic results (**concentration phenomenon**).
6- Improvement on recovery algorithms:
a) Reduced number of iterations needed for XH significantly;
b) Reduce the complexity of SBB from $O(n\log n)$ to $O(n)$.

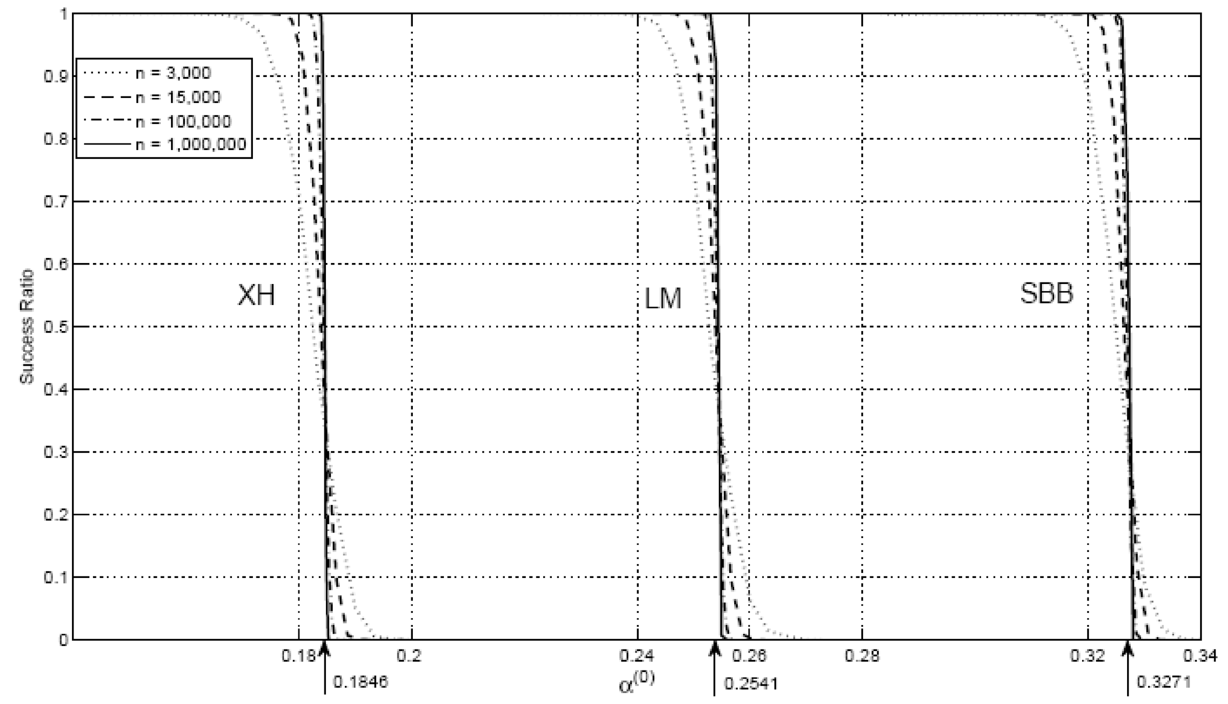


Average number of iterations for parallel and sequential XH algorithms. n = 100K. Theoretical thresholds are shown with arrows. Line style of the arrows represents the graph used.

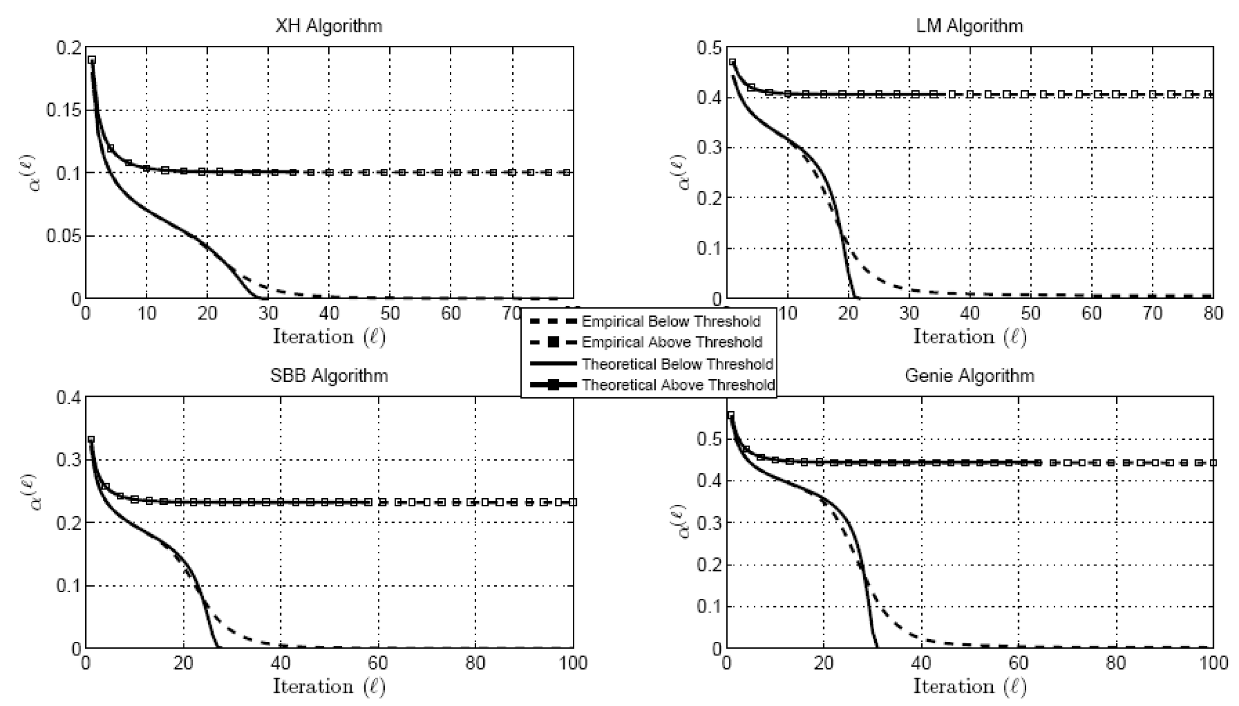
Success Threshold and Number of Iterations for Different Graphs and Algorithms

	(3,4)	(5,6)	(5,7)	(5,8)	(7,8)
XH	-	0.1846	0.1552	0.1339	0.1435
SBB	-	0.3271	0.2783	0.2421	0.3057
LM	0.2993	0.2541	0.2011	0.1646	0.2127
Genie	0.6474	0.5509	0.4786	0.4224	0.4708

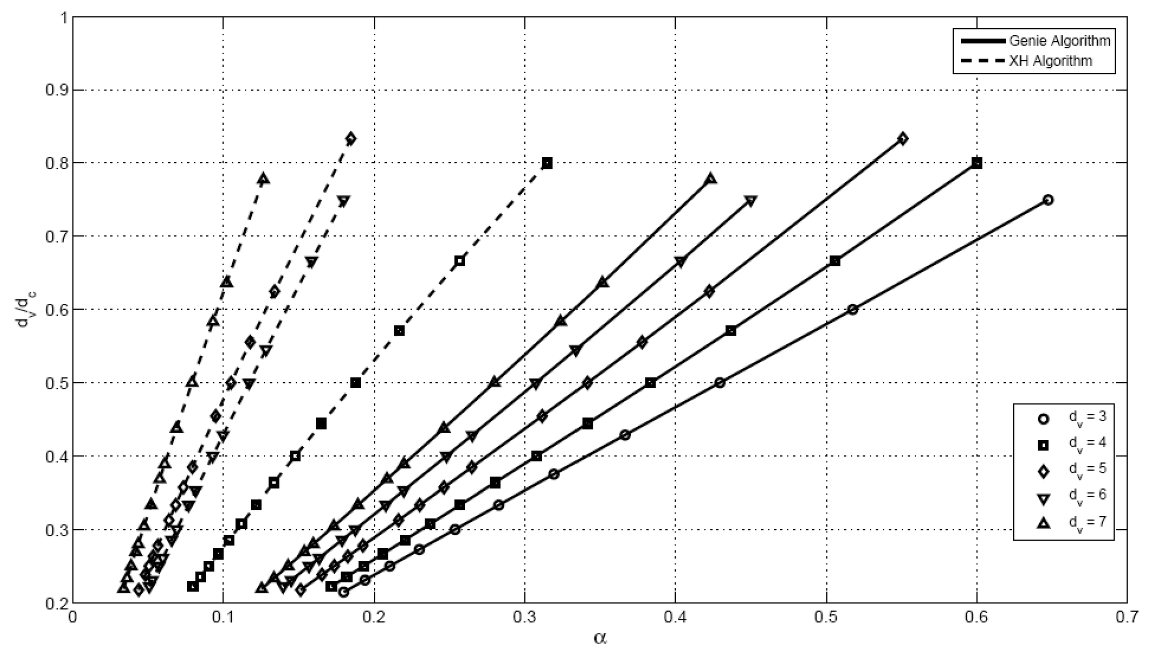
	(3,4)	(5,6)	(5,7)	(5,8)	(7,8)
XH	-	63	58	54	41
SBB	-	60	58	55	48
LM	82	42	37	33	27
Genie	106	66	66	62	55



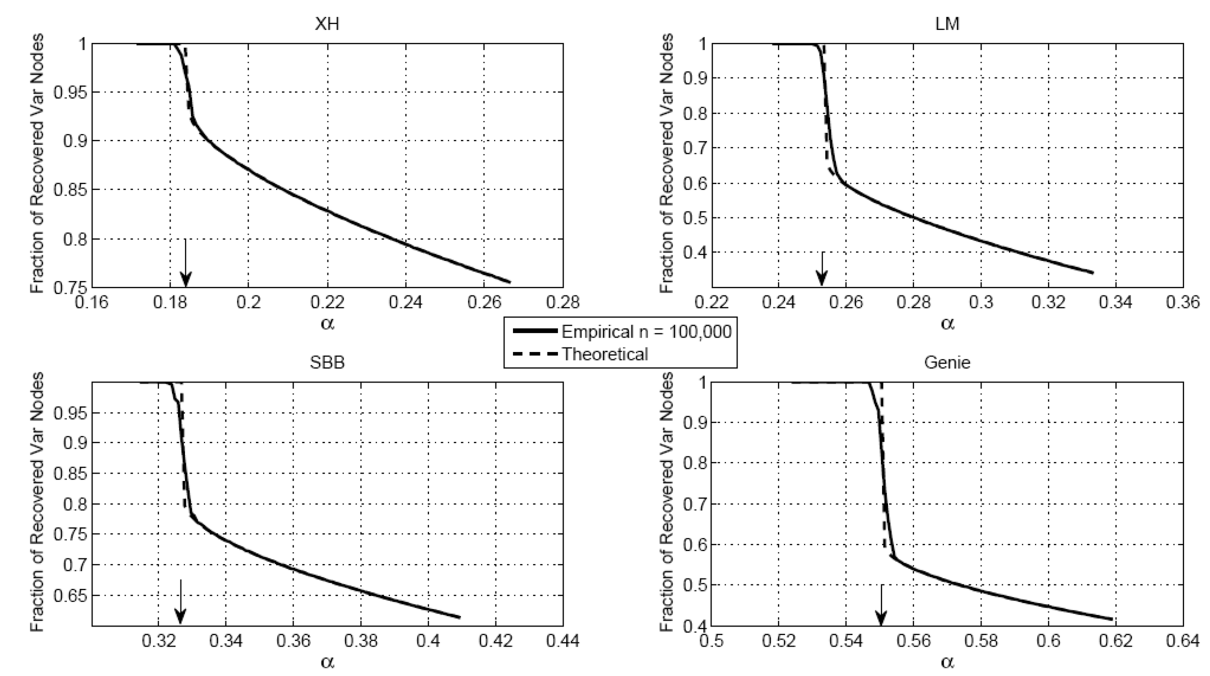
Success Ratio of Algorithms XH, LM and SBB for (5,6) graphs with n = 3K, 15K, 100K and 1000K. Analytical thresholds are shown by arrows.



Evolution of unresolved variable nodes vs. iteration number for the four recovery algorithms over a (5, 6) graph with n = 100K.



The slope represents the oversampling ratio. For each point the success threshold obtained from the theoretical analysis is reported. The two algorithms Genie and XH are compared.



Fraction of recovered variable nodes for different recovery algorithms. The graph used is a (5,6) random regular bipartite graph. For the simulation results, the length of the signal is chosen to be n = 100K. Each arrow represents the theoretical success threshold.

Selected References

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