



Evaluation of Marton's Inner Bound for the General Broadcast Channel

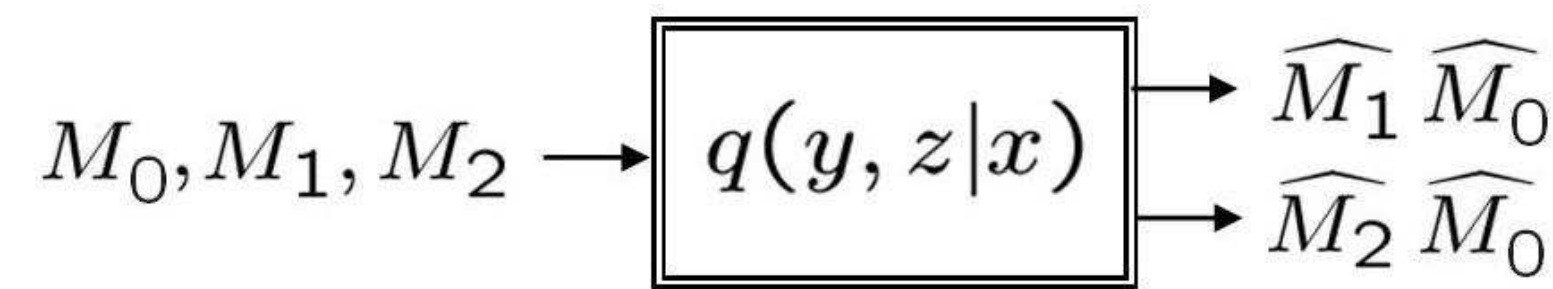
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Using a new tool for computing the second derivative of the entropy (under a certain perturbation), we prove cardinality bounds for the auxiliary random variables of Marton's inner bound.

Motivation

General broadcast channel:



Computing Marton's inner bound: Union over $p(u, v, w, x, y, z) = p(u, v, w, x)q(y, z|x)$

$$\begin{aligned} R_0, R_1, R_2 &\geq 0; \\ R_0 &\leq \min(I(W; Y), I(W; Z)); \\ R_0 + R_1 &\leq I(UW; Y); \\ R_0 + R_2 &\leq I(VW; Z); \\ R_0 + R_1 + R_2 &\leq I(U; Y|W) + I(V; Z|W) - I(U; V|W) \\ &\quad + \min(I(W; Y), I(W; Z)). \end{aligned}$$

Traditional use of the strengthened Carathéodory theorem of Fenchel and Eggleston gives:

$$\begin{aligned} |\mathcal{U}| &\leq |\mathcal{V}||\mathcal{X}| + 1 \\ |\mathcal{V}| &\leq |\mathcal{U}||\mathcal{X}| + 1 \end{aligned}$$

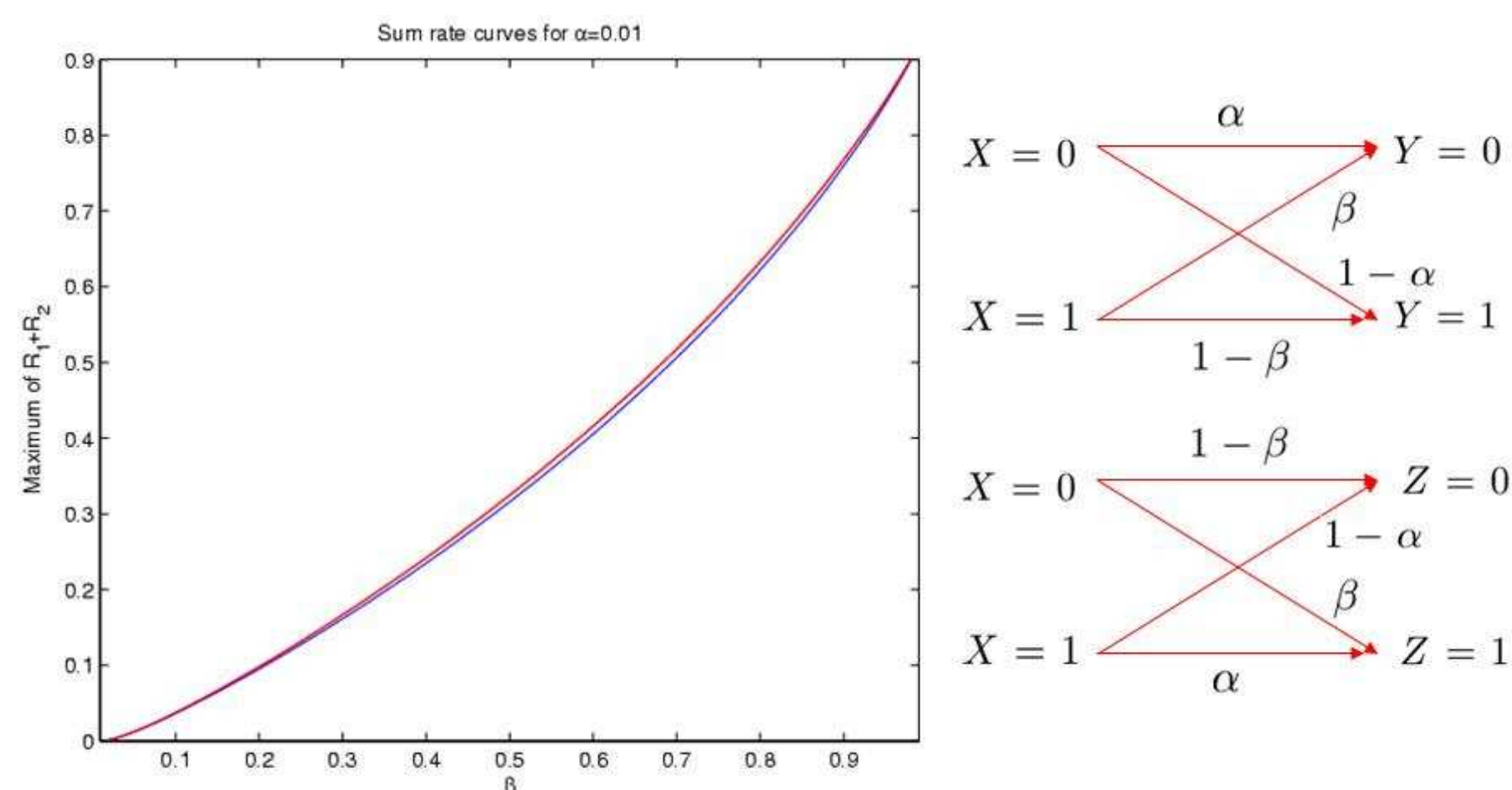
Does not lead to fixed cardinality bounds

Main Results

Cardinality bounds:

$$|\mathcal{U}| \leq |\mathcal{X}|, |\mathcal{V}| \leq |\mathcal{X}|, |\mathcal{W}| \leq |\mathcal{X}| + 4, H(X|UVW) = 0$$

Implication: gap between Marton's inner bound and Nair and El Gamal's outer bound



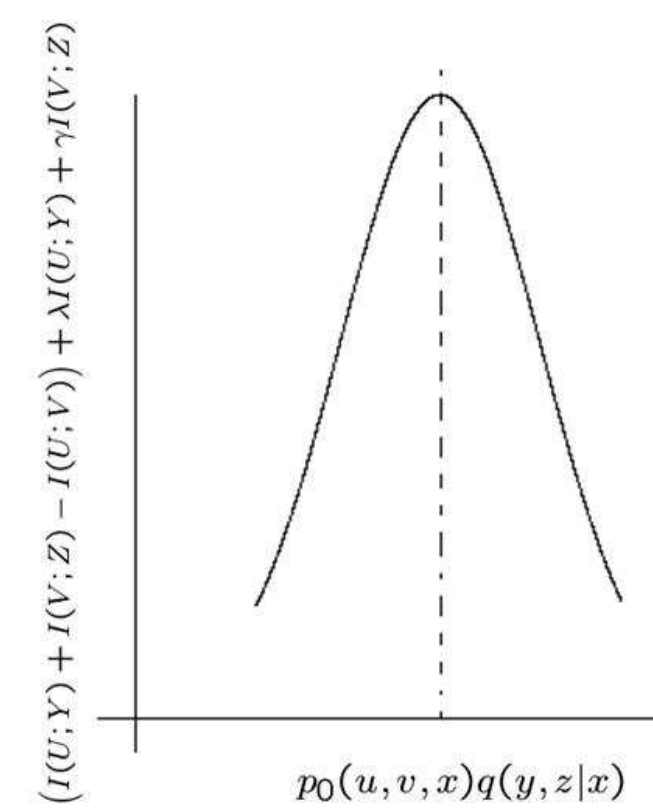
Proof scheme

- For simplicity and in order to show the main proof ingredients we consider finding cardinality bounds for

$$\max_{UV-X-YZ} I(U; Y) + I(V; Z) - I(U; V) + \lambda I(U; Y) + \gamma I(V; Z)$$

for given arbitrary $\lambda \geq 0, \gamma \geq 0$. Also assume $|\mathcal{U}| \leq K, |\mathcal{V}| \leq K$ for some K that can be arbitrarily large.

- For the simplified problem, we prove the sufficiency of $|\mathcal{U}| \leq |\mathcal{X}|, |\mathcal{V}| \leq |\mathcal{X}|$.
- Assume that a maximum occurs at some $p_0(u, v, x)q(y, z|x)$. For any perturbation



- The first derivative is zero; the second derivative is non-positive (messy to compute)
- We consider only perturbations that do not increase the support.

How to represent a perturbation

- Given $X_1, X_2, \dots, X_n \sim p_0(x_1, x_2, \dots, x_n)$, would like to define $p_\epsilon(x_1, x_2, \dots, x_n)$
- One choice: specify $[\frac{\partial}{\partial \epsilon} p_\epsilon](x_1, x_2, \dots, x_n)$ and work with joint distributions
 - Computing the first and second derivatives of various entropy terms is messy
 - Constraining the support to not increase is messy
- Better choice: specify the perturbation in proportion to $p_0(\cdot)$, via a random variable.
 - The elements of the derivative matrix are coded in a joint distribution

Perturbation and the Main Lemma for computing the second derivative

$X_1, X_2, \dots, X_n \sim p_0(x_1, x_2, \dots, x_n), L: \mathbb{E}[L] = 0, |\mathbb{E}[L|X_1 = x_1, \dots, X_n = x_n]| < \infty$

$$p_\epsilon(\widehat{X}_1 = x_1, \widehat{X}_2 = x_2, \dots, \widehat{X}_n = x_n) =$$

$$p_0(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \cdot (1 + \epsilon \mathbb{E}[L|X_1 = x_1, \dots, X_n = x_n]), \epsilon \in [-\epsilon_1, \epsilon_2]$$

Then $\forall I \subset \{1, 2, \dots, n\}$:

$$p_\epsilon(\widehat{X}_{i \in I} = x_{i \in I}) = p_0(X_{i \in I} = x_{i \in I}) \cdot (1 + \epsilon \mathbb{E}[L|X_{i \in I} = x_{i \in I}]);$$

$$\frac{\partial^2}{\partial \epsilon^2} H(\widehat{X}_{i \in I}) |_{\epsilon=0} = -\log(e) \cdot \mathbb{E}[\mathbb{E}[L|X_{i \in I}]^2];$$

$$\frac{\partial^2}{\partial \epsilon^2} H(\widehat{X}_{i \in I}) = -\log(e) \cdot I(\epsilon)$$

Defining a perturbation

$$\max_{UV-X-YZ, |\mathcal{U}| \leq K, |\mathcal{V}| \leq K} I(U; Y) + I(V; Z) - I(U; V) + \lambda I(U; Y) + \gamma I(V; Z)$$

Assume that a maximum occurs at $p_0(u, v, x)q(y, z|x)$.
Take $L: \mathcal{U} \times \mathcal{V} \times \mathcal{X} \rightarrow \mathbb{R}$ s.t. $\mathbb{E}[L(U, V, X)] = 0$, and let

$$\begin{aligned} p_\epsilon(\widehat{U} = u, \widehat{V} = v, \widehat{X} = x, \widehat{Y} = y, \widehat{Z} = z) = \\ p_0(U = u, V = v, X = x, Y = y, Z = z) \times \\ (1 + \epsilon \mathbb{E}[L(U, V, X)|U = u, V = v, X = x, Y = y, Z = z]) \end{aligned}$$

Or,

$$\begin{aligned} p_\epsilon(\widehat{U} = u, \widehat{V} = v, \widehat{X} = x, \widehat{Y} = y, \widehat{Z} = z) = \\ p_0(U = u, V = v, X = x, Y = y, Z = z) \times (1 + \epsilon L(u, v, x)) = \\ \underbrace{(p_0(U = u, V = v, X = x) \times (1 + \epsilon L(u, v, x)))}_{p_\epsilon(\widehat{U}=u, \widehat{V}=v, \widehat{X}=x)} \times \underbrace{q(\widehat{Y}=y, \widehat{Z}=z|\widehat{X}=x)} \end{aligned}$$

Therefore for any such L :

$$\frac{\partial^2}{\partial \epsilon^2} (I(\widehat{U}; \widehat{Y}) + I(\widehat{V}; \widehat{Z}) - I(\widehat{U}; \widehat{V}) + \lambda I(\widehat{U}; \widehat{Y}) + \gamma I(\widehat{V}; \widehat{Z})) |_{\epsilon=0} \leq 0$$

Assume $\mathbb{E}[L(U, V, X)|X] = 0$

$$p_\epsilon(\widehat{X} = x) = p_0(X = x) \times (1 + \epsilon \mathbb{E}[L(U, V, X)|X = x]) = p_0(X = x)$$

Implications of the second derivative being non-positive

$$\mathbb{E}[\mathbb{E}[L(U, V, X)|UY]^2] + \mathbb{E}[\mathbb{E}[L(U, V, X)|VZ]^2] - \mathbb{E}[\mathbb{E}[L(U, V, X)|UV]^2] \leq 0$$

Special case: $L': \mathcal{U} \rightarrow \mathbb{R}$ such that $\mathbb{E}[L'(U)|X] = 0$

$$\mathbb{E}[\mathbb{E}[L'(U)|UY]^2] = \mathbb{E}[\mathbb{E}[L'(U)|VZ]^2] = \mathbb{E}[L'(U)^2]$$

Therefore

$$\forall L': \mathcal{U} \rightarrow \mathbb{R} \text{ s.t. } \mathbb{E}[L'(U)|X] = 0, \mathbb{E}[L'(U)|VZ] = 0$$

It is then shown that any perturbation $p_\epsilon(\widehat{U} = u, \widehat{V} = v, \widehat{X} = x, \widehat{Y} = y, \widehat{Z} = z)$ that does not increase the support, where

$$\begin{aligned} p_\epsilon(\widehat{V} = v, \widehat{X} = x, \widehat{Y} = y, \widehat{Z} = z | \widehat{U} = u) = p_0(V = v, X = x, Y = y, Z = z | U = u) \\ \text{and} \\ p_\epsilon(\widehat{X} = x) = p_0(X = x) \end{aligned}$$

implies

$$p_\epsilon(\widehat{V} = v, \widehat{Z} = z) = p_0(V = v, Z = z)$$