Fast Algorithms and Performance Bounds for Sum Rate Maximization in Wireless Networks

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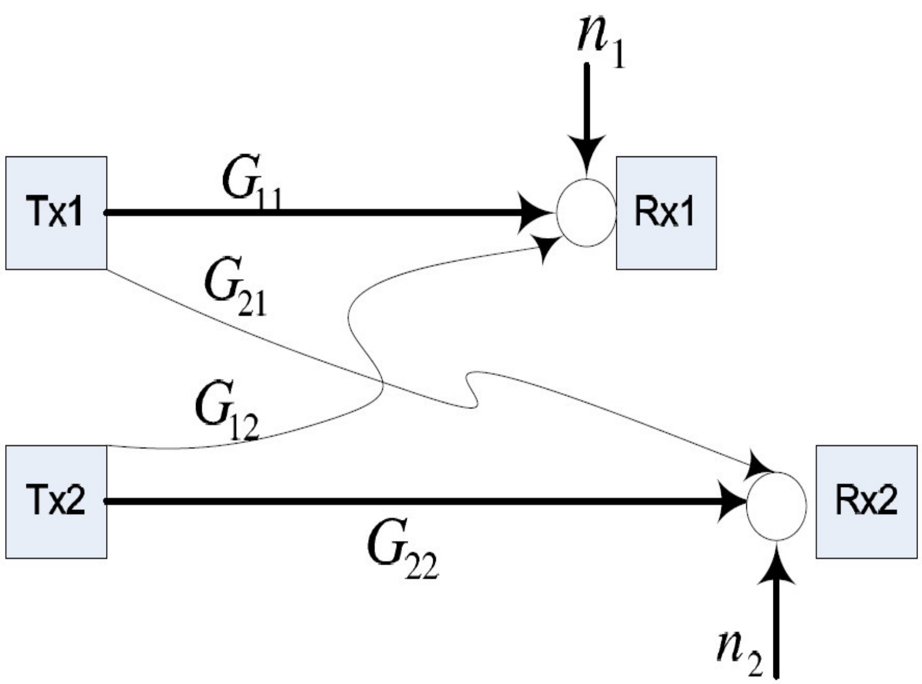
12th Aug 2009

System Model

- Interference channel: Treat interference as additive Gaussian noise



Performance Metric

- Signal-to-Interference Ratio:

$$\mathsf{SIR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}$$

with G_{lj} the channel gains from transmitter j to receiver l and n_l the additive white Gaussian noise (AWGN) power at receiver l

- Attainable data rate (nats per channel use) is a function of SIR, e.g., Shannon capacity formula $r_l = \log(1 + \mathsf{SIR}_l)$

- Power constraints $\mathbf{p} \leq \bar{\mathbf{p}}$

Interference Parameters

- Let $\mathbf{F}$ be a nonnegative matrix with entries:

$$F_{ij} = \begin{cases} 0, & \text{if } i = j \\ \frac{G_{ij}}{G_{ii}}, & \text{if } i \neq j \end{cases}$$

and

$$\mathbf{v} = \left(\frac{n_1}{G_{11}}, \frac{n_2}{G_{22}}, \ldots, \frac{n_L}{G_{LL}} \right)^\top .$$

Problem: Maximize Sum Shannon Rates

- Find $\mathbf{p}^\star = \arg \max_{\mathbf{0} \le \mathbf{p} \le \bar{\mathbf{p}}} \sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}))$ where $\mathbf{1}^\top \mathbf{w} = 1$

- Characterize the achievable rate region: $r_l = \log(1 + \mathsf{SIR}_l(\mathbf{p})) \; \forall \, l$

- Two-User case:

$$\max \quad w_1 \log\left(1 + \frac{G_{11} p_1}{G_{12} p_2 + n_1}\right) + w_2 \log\left(1 + \frac{G_{22} p_2}{G_{21} p_1 + n_2}\right)$$

subject to: $\quad 0 \le p_1 \le \bar{p}_1, \;\; 0 \le p_2 \le \bar{p}_2$

Weighted Max-Min SIR

- Consider $\max_{\mathbf{p}\geq\mathbf{0}} \min_l \frac{\mathsf{SIR}_l(\mathbf{p})}{\beta_l}$ subject to $p_l \leq \bar{p}_l \quad \forall\, l$

- **Theorem 1.** *The optimal solution is such that the value* SIR_l/β_l *for all users are equal. The optimal weighted max-min* SIR *is given by*

$$\gamma^* = \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_i)\mathbf{v}\mathbf{e}_i^\top))},$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))}.$$

Further, all links i *transmit at peak power* $\bar{p}_i$ *and the rest do not. Further, the optimal* $\mathbf{p}$*, denoted by* $\mathbf{p}^*$*, is* $t\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p}_l)\mathbf{v}\mathbf{e}_l^\top))$ *for some constant* $t > 0$.

Max-min SIR: Primal-Dual Algorithm

- **Algorithm 1. [Weighted Max-min** SIR **Algorithm]**

  1. *Initialize an arbitrarily positive* $\mathbf{w}(t)$ *and small* $\epsilon, \alpha(1)$*.*
  2. *Set* $\mathbf{p}(0) = \bar{\mathbf{p}}$*. Repeat*

  $$p_l(k+1) = \min\left\{ w_l(t) / \left(\sum_{j \neq l} \frac{w_j(t) F_{jl} \mathsf{SIR}_j(\mathbf{p}(k))}{p_j(k)} \right), \bar{p}_l \right\}$$

  until $\|\mathbf{p}(k+1) - \mathbf{p}(k)\| \leq \epsilon$*.*

  3. *Compute*

  $$w_l(t+1) = \max\{w_l(t) \quad + \alpha(t)(\textstyle\sum_j w_j(t) \log(\mathsf{SIR}_j(\mathbf{p}(k+1))/\beta_j) - \log(\mathsf{SIR}_l(\mathbf{p}(k+1))/\beta_l)), \, 0\}$$

  for all l*, where* t *indexes discrete time slots much larger than* k*.*

  4. *Normalize* $\mathbf{w}(t+1)$ *so that* $\mathbf{1}^\top \mathbf{w}(t+1) = 1$*. Go to Step 2.*

A Faster Max-min SIR Algorithm

- Based on Nonlinear Perron-Frobenius Theory

- **Algorithm 2. [Equal power constrained Max-min** SIR**]**

  1. *Update power* $\mathbf{p}(k+1)$*:*

  $$p_l(k+1) = \frac{\beta_l}{\mathsf{SIR}_l(\mathbf{p}(k))} p_l(k) \;\; \forall\, l.$$

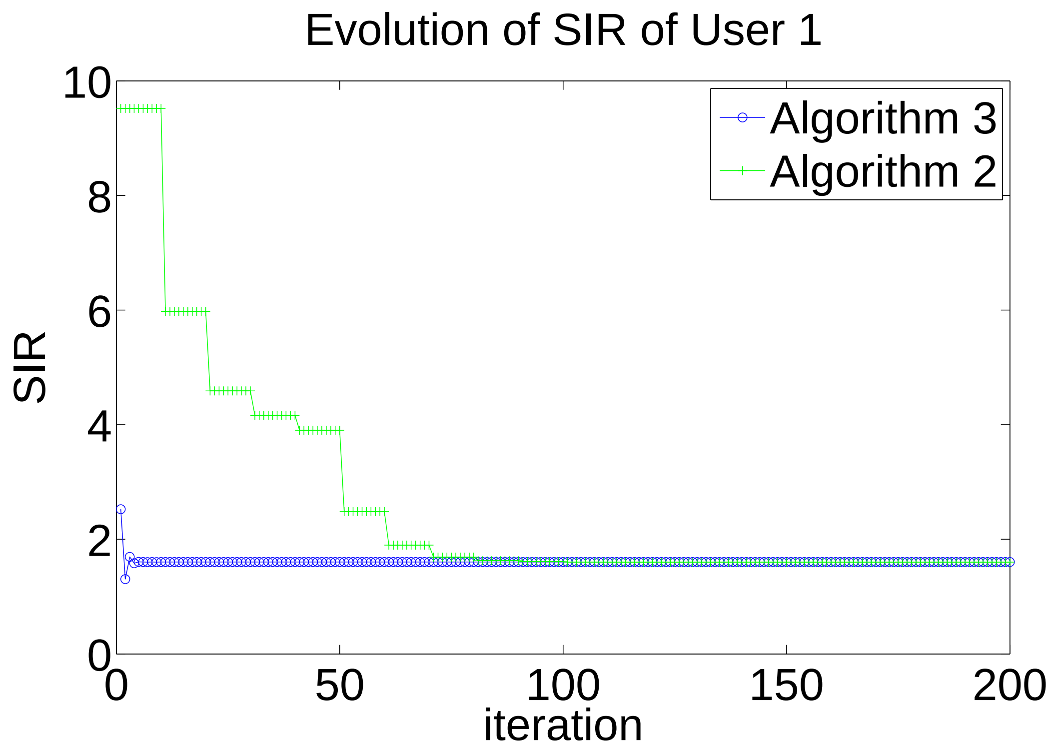
  2. *Normalize* $\mathbf{p}(k+1)$*:*

  $$p_l(k+1) = p_l(k+1) / \max_j p_j(k+1) \cdot \bar{p} \;\; \forall\, l.$$

- **Theorem 2.** *Starting from any initial point* $\mathbf{p}(0)$, $\mathbf{p}(k)$ *in Algorithm 2 converges geometrically fast to* $\mathbf{x}(\mathrm{diag}(\boldsymbol{\beta})(\mathbf{F} + (1/\bar{p})\mathbf{v}\mathbf{e}_i^\top))$ *(unique up to a scaling constant).*

Max-min SIR: Example

- The nonlinear Perron-Frobenius theory based algorithm is much faster than the subgradient algorithm



Fast & Simple Algorithms

Quasi-Inverse of Nonnegative Matrices

- Definition **[Wong54]**: $\mathbf{B}$ is a quasi-inverse of $\tilde{\mathbf{B}} \geq \mathbf{0}$ if $\mathbf{B} - \tilde{\mathbf{B}} = \mathbf{B}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\mathbf{B} \geq \mathbf{0}$

- $\rho(\tilde{\mathbf{B}}) = \frac{\rho(\mathbf{B})}{1+\rho(\mathbf{B})}$

- $\mathbf{x}(\tilde{\mathbf{B}}) = \mathbf{x}(\mathbf{B})$ & $\mathbf{y}(\tilde{\mathbf{B}}) = \mathbf{y}(\mathbf{B})$

Interference & SNR Regime

- Consider the matrix

$$\mathbf{B} = \mathbf{F} + \sum_l \frac{1}{\mathbf{1}^\top \overline{\mathbf{p}}} \mathbf{v} \mathbf{e}_l^\top$$

- (High SNR regime) $\tilde{\mathbf{B}}$ does not exist

  or any nonnegative matrix with a zero trace & positive off-diagonals

- (Low SNR regime) $\tilde{\mathbf{B}}$ always exists

  or any nonnegative matrix that is a dyad

- (Low interference/moderate SNR regime) $\tilde{\mathbf{B}}$ almost always exists

Tight Upper Bound: Key Theorem

- If $\tilde{\mathrm{B}} \geq \mathbf{0}$, then

$$\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right),$$

where $\mathbf{x}, \mathbf{y}$ are the Perron and left eigenvectors of $\mathbf{B}$ respectively.

- Main ideas of proof:
  - Quasi-invertibility of nonnegative matrix **[Wong54]**
  - Friedland-Karlin Inequalities **[FriedlandKarlin75]**

- Physical meaning of upper bound

Physical Interpretation of Upper Bound (I)

- $\sum_l w_l \log(1 + \mathsf{SIR}_l(\mathbf{p}^\star)) \leq \|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}} \log\left(1 + 1/\rho(\mathbf{B})\right).$

- ▪ $\|\mathbf{w}\|_\infty^{\mathbf{x}\circ\mathbf{y}}$ as an approximation ratio using

$$\begin{array}{ll} \text{maximize} & \min_l \mathsf{SIR}_l(\mathbf{p}) \\ \text{subject to} & \mathbf{1}^\top \mathbf{p} \leq \mathbf{1}^\top \overline{\mathbf{p}} \\ \text{variables:} & \mathbf{p}. \end{array}$$

▪ Closed-form solution (via Nonlinear Perron-Frobenius Theory):

$$\text{Optimal solution}: \quad 1/\rho(\mathbf{B}), \quad \mathbf{B} = \mathbf{F} + (1/\mathbf{1}^\top \bar{\mathbf{p}})\mathbf{v}\mathbf{1}^\top;$$

$$\text{Optimizer}: \quad \mathbf{x}(\mathbf{B})$$

Performance Guarantee: Weighted Max-min SIR

- **Theorem 3.** *Suppose* $\tilde{\mathbf{B}} \geq \mathbf{0}$*. Let*

$$\eta = \frac{\sum_l w_l \log(1 + w_l/\rho(\mathit{diag}(\mathbf{w})(\mathbf{F} + (1/\overline{p}_i)\mathbf{v}\mathbf{e}_i^\top)))}{\|\mathbf{w}\|_\infty^{\mathbf{x}(\mathbf{B})\circ\mathbf{y}(\mathbf{B})} \log(1 + 1/\rho(\mathbf{B}))},$$

where

$$i = \arg\min_l \frac{1}{\rho(\mathit{diag}(\mathbf{w})(\mathbf{F} + (1/\overline{p}_l)\mathbf{v}\mathbf{e}_l^\top))}.$$

Then, η *is an approximation ratio by solving the constrained max-min weighted SIR problem:*

$$\begin{array}{rl} \textit{maximize} & \min_l \dfrac{\mathsf{SIR}_l(\mathbf{p})}{w_l} \\ \textit{subject to} & \mathbf{p} \leq \overline{\mathbf{p}} \\ \textit{variables:} & \mathbf{p}. \end{array}$$

Quasi-invertibility in Wireless Network: Example

Parameter	Avg. % of $\tilde{\mathbf{B}} \geq \mathbf{0}$	SAPC (η)	Max-min SIR (η)	On-off sched. (η)
$\bar{p}_l = 33$mW $\forall l$ SNR $= 7$dB	99	0.97 (0.93)	0.99 (0.96)	0.89 (0.84)
$\bar{p}_l = 1$W $\forall l$ SNR $= 40$dB	65	0.87 (0.82)	0.91 (0.83)	0.87 (0.82)

Table 1: A typical numerical example in a ten-user network with two different maximum power constraint settings.

Provably Good Near-optimal Fast Algorithms